

Multiagent Systems and Games

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Lecture 6

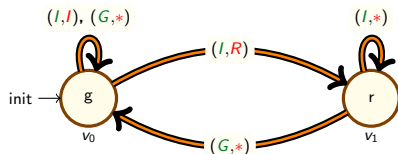
Definition

A Multiagent System is a tuple $\mathcal{M} = \langle \mathcal{AP}, \text{Ag}, (\text{Act}_i)_{i \in \text{Ag}}, V, v_0, \tau, E \rangle$ where

- $\text{Ag} = \{0, 1, \dots, k\}$ is the set of agents
- Act_i is the set of actions of Agent $i \in \text{Ag}$
- V is the set of states in the system
- v_0 is the initial state
- $\tau : V \rightarrow 2^{\mathcal{AP}}$ is the labeling function
- $E : V \times \text{Act}_0 \times \text{Act}_1 \times \dots \times \text{Act}_k \rightarrow V$ is the transition function

Multiagent System - Example

- Let a Multiagent System with two components : an user(sends requests) and a server (grants requests)
- Agents: 0 - server, 1 - user
- States: $V = \{v_0, v_1\}$
- Labels: $\tau(v_0) = \{g\}$ (all requests are granted), $\tau(v_1) = \{r\}$ (there is at least one request unsolved)
- Actions: $Act_0 = \{I, G\}$, $Act_1 = \{I, R\}$



Definition

A **concurrent game** is a tuple $\mathcal{G} = \langle \mathcal{M}, (\Theta_i)_{i \in \text{Ag}} \rangle$ where

- $\mathcal{M} = \langle \mathcal{AP}, \text{Ag}, (\text{Act}_i)_{i \in \text{Ag}}, V, v_0, \tau, E \rangle$ is a multiagent system
- $\Theta_i \subseteq V^\omega$ is the objective of Agent $i \in \text{Ag}$

- A **play** in \mathcal{G} is a sequence $\pi = v_0 v_1 v_2 \dots$ such that

$$\forall n \geq 0, \exists (a_0^n, a_1^n, \dots, a_k^n) \text{ s.t. } (v_n, (a_0^n, a_1^n, \dots, a_k^n), v_{n+1}) \in E$$

- A **history** is a finite prefix of a play.
- Agent i **wins** (satisfies its objective) along a play π if $\pi \in \Theta_i$

Strategies

- Given a multiagent system $\mathcal{M} = \langle \mathcal{AP}, \text{Ag}, (\text{Act}_i)_{i \in \text{Ag}}, V, v_0, \tau, E \rangle$,
- A **strategy for Agent i** is a mapping $\sigma_i : V(\text{Act}_i V)^* \rightarrow \text{Act}_i$
Agent i sees only the states and his actions!
- A **strategy profile** is a tuple $\bar{\sigma} = (\sigma_0, \dots, \sigma_k)$ of strategies, one for each agent
- A play $\pi = v_0 v_1 v_2 \dots$ is **compatible** with a strategy σ_i of Agent i if $\exists (a_0^0, a_1^0, \dots, a_k^0) (a_1^1, a_2^1, \dots, a_k^1) \dots$ s.t. $\forall n \geq 0$,
 - $v_{n+1} = E(v_n, (a_0^n, a_1^n, \dots, a_k^n))$ and
 - $a_i^n = \sigma_i(v_0 a_0^0 v_1 a_1^1 v_2 \dots v_n)$
- $\text{out}(\sigma_i)$ is the set of plays compatible with σ_i
- $\text{out}(\bar{\sigma}) = \bigcap_{i \in \text{Ag}} \text{out}(\sigma_i)$ is the **outcome** of the profile $\bar{\sigma}$ - only one play!

- Given a concurrent game $\mathcal{G} = \langle \mathcal{M}, (\Theta_i)_{i \in \text{Ag}} \rangle$,
- A strategy σ_i of Agent i is **winning** from $v \in V$ if for any play π that starts in v and is compatible with σ_i , $\pi \in \Theta_i$.
- The **winning region** of Agent i is the set $W_i(\mathcal{G}) \subseteq V$ of all vertices v s.t.
Agent i has a winning strategy from v in \mathcal{G} .

Particular Winning Conditions

Let $\pi = v_0 v_1 v_2 \dots \in V^\omega$.

$\text{visit}(\pi) = \{v \mid v_\ell = v \text{ for some } \ell \geq 0\}$ and

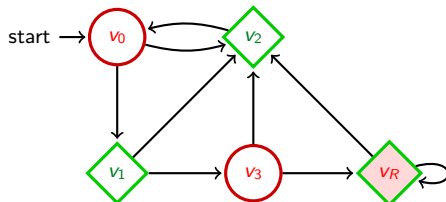
$\text{inf}(v_0 v_1 v_2 \dots) = \{v \mid v_\ell = v \text{ for infinitely many } \ell \geq 0\}$

A winning condition is a set $\Theta_i \subseteq V^\omega$. Particular cases:

- *Reachability*: For a set $R \subseteq V$, $\text{Reach}(R) = \{\pi \in V^\omega \mid \text{visit}(\pi) \cap R \neq \emptyset\}$
- *Safety*: For a set $S \subseteq V$, $\text{Safe}(S) = \{\pi \in V^\omega \mid \text{visit}(\pi) \subseteq S\}$
- *Büchi*: For a set $T \subseteq V$, $\text{Buchi}(T) = \{\pi \in V^\omega \mid \text{inf}(\pi) \cap T \neq \emptyset\}$
- *coBüchi*: For a set $T \subseteq V$, $\text{coBuchi}(T) = \{\pi \in V^\omega \mid \text{inf}(\pi) \cap T = \emptyset\}$
- *Parity*: For a priority function $p : V \rightarrow \mathbb{N}$,
 $\text{Parity}(p) = \{\pi \in V^\omega \mid \min\{p(v) \mid v \in \text{inf}(\pi)\} \text{ is even}\}$
- *LTL*: For an LTL formula φ , $\text{LTL}(\varphi) = \{\pi \in V^\omega \mid \text{trace}(\pi), 0 \models \varphi\}$

Two-players Zero-sum Turn-based Games

- Play only **two players** in **turns** and choose the next state
- Zero-sum : when **one player wins, the other loses**



- A Two-players Zero-sum Turn-based Games is a tuple $\mathcal{G} = \langle \text{Ag} = \{A, B\}, V = V_A \uplus V_B, v_0, E, \Theta_A \rangle$ where
 - A is the protagonist and B is the adversary
 - V_A : the states controlled by Player A
 - V_B : the states controlled by Player B
 - $E \subseteq V \times V$
 - Θ_A : the objective of Player A
 - **Player B wants to avoid Θ_A**
- A **strategy** for Player A is a mapping $\sigma_A : V^* V_A \rightarrow V$

Games: Fundamental Questions

Determinacy: Given a class Γ of games, is it the case that for any game $\mathcal{G} \in \Gamma$, either Player A or Player B has a winning strategy? I.e.,

Does there exist a partition $V = W_A(\mathcal{G}) \uplus W_B(\mathcal{G})$?

Decision: Given a game \mathcal{G} , does Player A (or Player B) have a winning strategy? I.e.,

Is the case that $v_0 \subseteq W_A(\mathcal{G})$ (or $v_0 \subseteq W_B(\mathcal{G})$) ?

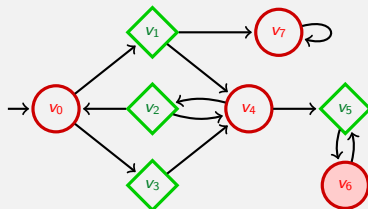
Construction: Given a game \mathcal{G} , how to construct a winning strategy?

Complexity: Can a winning strategy be implemented algorithmically? What are the needed resources?

Turn-based Zero-sum Reachability Games

REACHABILITY GAME:

$$\mathcal{G} = \langle V = V_A \uplus V_B, v_0, E, \text{REACH}(R) \rangle \text{ for } R \subseteq V$$



- Winning condition for Player A :
- Solution complexity:
- Algorithm:
- Dual game:

$$\text{visit}(\pi) \cap R \neq \emptyset$$

linear time in $|E|$

attractors

Safety

Turn-based Zero-sum Reachability Games : Attractors

- Let $E(v) = \{w \in V \mid (v, w) \in E\}$
- Inductively construct the set $Attr_i^A(R)$ from which Player A can force to reach R in at most i steps

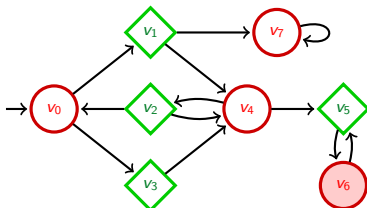
$$Attr_0^A(R) := R$$

$$Attr_{n+1}^A(R) := Attr_n^A(R) \cup \\ \cup \{v \in V_A \mid E(v) \cap Attr_n^A(R) \neq \emptyset\} \\ \cup \{v \in V_B \mid E(v) \subseteq Attr_n^A(R)\}$$

- $Attr_n^A(R)$ increases until reaching the fixed point $Attr_{n+1}^A(R) = Attr_n^A(R)$.
- Let $Attr_A(R)$ be the fixed point and call it **attractor of R for Player A**

Turn-based Zero-sum Reachability Games : Attractor Example

Example



Player A : ○

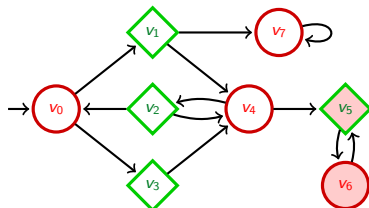
Player B : ◇

$R = \{v_6\}$

$$\text{Attr}_0^A(\{v_6\}) = \{v_6\}$$

Turn-based Zero-sum Reachability Games : Attractor Example

Example



Player A : ○

Player B : ◇

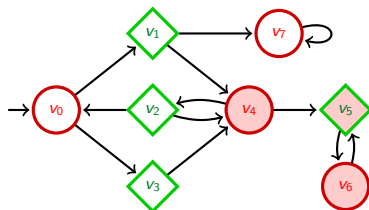
$R = \{v_6\}$

$$\text{Attr}_0^A(\{v_6\}) = \{v_6\}$$

$$\text{Attr}_1^A(\{v_6\}) = \{v_6, v_5\}$$

Turn-based Zero-sum Reachability Games : Attractor Example

Example



Player A : ○

Player B : ◇

$R = \{v_6\}$

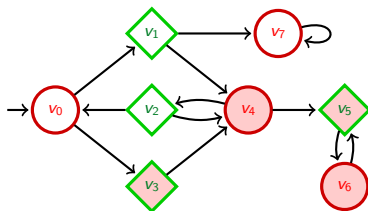
$$\text{Attr}_0^A(\{v_6\}) = \{v_6\}$$

$$\text{Attr}_1^A(\{v_6\}) = \{v_6, v_5\}$$

$$\text{Attr}_2^A(\{v_6\}) = \{v_6, v_5, v_4\}$$

Turn-based Zero-sum Reachability Games : Attractor Example

Example



Player A : ○

Player B : ◇

$R = \{v_6\}$

$$\text{Attr}_0^A(\{v_6\}) = \{v_6\}$$

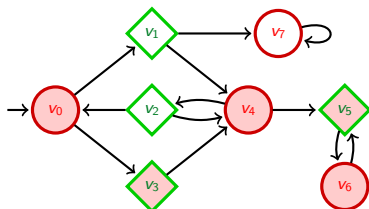
$$\text{Attr}_1^A(\{v_6\}) = \{v_6, v_5\}$$

$$\text{Attr}_2^A(\{v_6\}) = \{v_6, v_5, v_4\}$$

$$\text{Attr}_3^A(\{v_6\}) = \{v_6, v_5, v_4, v_3\}$$

Turn-based Zero-sum Reachability Games : Attractor Example

Example



Player A : ○

Player B : ◇

$R = \{v_6\}$

$$\text{Attr}_0^A(\{v_6\}) = \{v_6\}$$

$$\text{Attr}_1^A(\{v_6\}) = \{v_6, v_5\}$$

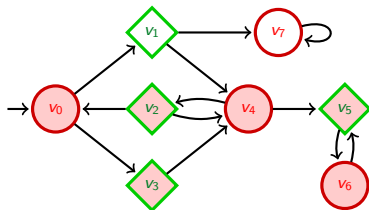
$$\text{Attr}_2^A(\{v_6\}) = \{v_6, v_5, v_4\}$$

$$\text{Attr}_3^A(\{v_6\}) = \{v_6, v_5, v_4, v_3\}$$

$$\text{Attr}_4^A(\{v_6\}) = \{v_6, v_5, v_4, v_3, v_0\}$$

Turn-based Zero-sum Reachability Games : Attractor Example

Example



Player A : ○

Player B : ◇

$R = \{v_6\}$

$$\text{Attr}_0^A(\{v_6\}) = \{v_6\}$$

$$\text{Attr}_1^A(\{v_6\}) = \{v_6, v_5\}$$

$$\text{Attr}_2^A(\{v_6\}) = \{v_6, v_5, v_4\}$$

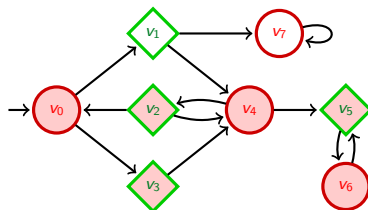
$$\text{Attr}_3^A(\{v_6\}) = \{v_6, v_5, v_4, v_3\}$$

$$\text{Attr}_4^A(\{v_6\}) = \{v_6, v_5, v_4, v_3, v_0\}$$

$$\text{Attr}_5^A(\{v_6\}) = \{v_6, v_5, v_4, v_3, v_0, v_2\}$$

Turn-based Zero-sum Reachability Games : Attractor Example

Example



Player A : ○

Player B : ◇

$R = \{v_6\}$

$$\text{Attr}_0^A(\{v_6\}) = \{v_6\}$$

$$\text{Attr}_1^A(\{v_6\}) = \{v_6, v_5\}$$

$$\text{Attr}_2^A(\{v_6\}) = \{v_6, v_5, v_4\}$$

$$\text{Attr}_3^A(\{v_6\}) = \{v_6, v_5, v_4, v_3\}$$

$$\text{Attr}_4^A(\{v_6\}) = \{v_6, v_5, v_4, v_3, v_0\}$$

$$\text{Attr}_5^A(\{v_6\}) = \{v_6, v_5, v_4, v_3, v_0, v_2\}$$

$$\text{Attr}_6^A(\{v_6\}) = \{v_6, v_5, v_4, v_3, v_0, v_2\} = \text{Attr}^A(\{v_6\})$$

Turn-based Zero-sum Reachability Games: Winning Regions and Strategies

Given a reachability game \mathcal{G} where Player A targets $R \subseteq V$,

- Let $rank : V \rightarrow \mathbb{N} \cup \{\infty\}$ s.t.

$$rank(v) = \begin{cases} \min\{n \mid v \in Attr_n^A(R)\} & \text{if } v \in Attr^A(R) \\ \infty & \text{otherwise} \end{cases}$$

- for every $v \in Attr^A$, either
 - $v \in R$
 - $v \in V_A \setminus R$ and $\exists w \in E(v)$ s.t. $rank(w) < rank(v)$, or
 - $v \in V_B \setminus R$ and $\forall w \in E(v)$ s.t. $rank(w) < rank(v)$
- for every $v \notin Attr^A$, either
 - $v \in V_A$ and $\forall w \in E(v)$, holds $rank(w) = \infty$, or
 - $v \in V_B$ and $\exists w \in E(v)$, holds $rank(w) = \infty$, or

Turn-based Zero-sum Reachability Games: Winning Regions and Strategies

Theorem

In a reachability game \mathcal{G} where Player A targets $R \subseteq V$, the winning region of Player A is

$$W_A(\mathcal{G}) = \text{Attr}^A(R)$$

Proof.

Let $v_0 \in \text{Attr}^A(R)$.

We construct a strategy $\sigma_A : V^* V_A \rightarrow V$ for Player A s.t. $\forall v_0 v_1 \dots v_\ell \in V^* V_A$,

$$\sigma_A(v_0 v_1 \dots v_\ell) = w \text{ s.t. } w \in E(v_\ell) \text{ and } \text{rank}(w) < \text{rank}(v_\ell)$$

Then, for any infinite play $v_0 v_1 v_2 \dots \in \text{out}(\sigma_A)$, holds

$$\text{rank}(v_0) > \text{rank}(v_1) > \text{rank}(v_2) > \dots$$

Therefore, we reach $\text{rank}(v_\ell) = 0$ which means $v_\ell \in R \Rightarrow \sigma_A$ is winning for Player A □

Turn-based Zero-sum Reachability Games: Winning Regions and Strategies

Theorem

In a reachability game \mathcal{G} where Player A targets $R \subseteq V$, the winning region of Player B is

$$W_B(\mathcal{G}) = V \setminus \text{Attr}^A(R)$$

Proof.

Let $v_0 \in V \setminus \text{Attr}^A(R)$.

We construct a strategy $\sigma_B : V^* V_B \rightarrow V$ for Player B s.t. $\forall v_0 v_1 \dots v_\ell \in V^* V_B$,

$$\sigma_A(v_0 v_1 \dots v_\ell) = w \text{ s.t. } w \in E(v_\ell) \text{ and } \text{rank}(w) = \infty$$

Then, for any infinite play $v_0 v_1 v_2 \dots \in \text{out}(\sigma_A)$, holds

$$\text{rank}(v_\ell) = \infty \quad \forall \ell \geq 0$$

Therefore, we never reach $\text{rank}(v_\ell) = 0 \Rightarrow v_\ell \notin R \quad \forall \ell \geq 0 \Rightarrow \sigma_B$ is winning for Player B □

Turn-Based Zero-sum Reachability Games

Theorem

There is a linear-time algorithm to solve Reachability games.

Algorithm 1: A linear-time algorithm for solving reachability games

Input: a game $G = (V, E, V_0, F)$ with reachability condition F

Output: winning regions W_0 and W_1

```
// initialisation
forall the  $v \in V$  do
  | win[ $v$ ] :=  $\perp$ 
  |  $P[v]$  :=  $\emptyset$ 
  |  $n[v]$  := 0
end
// for each position, set up a list of predecessors and count its successors
forall the  $(u, v) \in E$  do
  |  $P[v]$  :=  $P[v] \cup \{u\}$ 
  |  $n[u]$  :=  $n[u] + 1$ 
end
forall the  $v \in F$  do
  | Propagate( $v$ )
end
return win
```

```
procedure Propagate( $v$ )
if win[ $v$ ]  $\neq \perp$  then return
// mark  $v$  as winning for Player 0
win[ $v$ ] :=  $\top$ 
// propagate change to predecessors
forall the  $u \in P[v]$  do
  |  $n[u]$  :=  $n[u] - 1$ 
  | if  $u \in V_0$  or  $n[u] = 0$  then Propagate( $u$ )
end
```

Concurrent Multiplayer Reachability Games

The Game: $\mathcal{G} = \langle \mathcal{AP}, \text{Ag}, (\text{Act}_i)_{i \in \text{Ag}}, V, v_0, \tau, E, (\text{reach}(R_i))_{i \in \text{Ag}} \rangle$

Decision Problem: Given the concurrent game \mathcal{G} , does Player i have a winning strategy?

- Recall: Each Player i has a target set R_i !
- Let $\text{Succ}(v, a_i) = \{w \in V \mid \exists (a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_k) \text{ s.t. } w = E(v, \bar{a})\}$
- Compute the attractor using the formula:

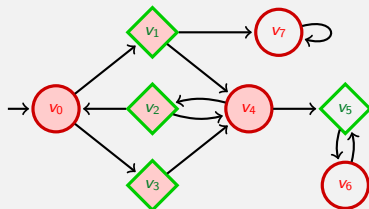
$$\begin{aligned} \text{Attr}_0^i(R_i) &:= R_i \\ \text{Attr}_{n+1}^i(R_i) &:= \text{Attr}_n^i(R_i) \cup \\ &\quad \cup \{v \in V \mid \exists a_i \in \text{Act}_i \text{ s.t. } \text{Succ}(v, a_i) \subseteq \text{Attr}_n^i(R_i)\} \end{aligned}$$

- Let $\text{Attr}^i(R_i)$ be the fixed point where $\text{Attr}_{n+1}^i(R_i) := \text{Attr}_n^i(R_i)$
- The winning region of Player i is $W_i(\mathcal{G}) = \text{Attr}^i(R_i)$

Turn-based Zero-Sum Safety Game

SAFETY GAME:

$$\mathcal{G} = \langle V = V_A \uplus V_B, v_0, E, \text{SAFE}(S) \rangle \text{ for } S \subseteq V$$



- Winning condition for Player A :
- Solution complexity:
- Algorithm:
- Dual game:

$$\text{visit}(\pi) \subseteq S$$

linear time in $|E|$

dualize + attractors

Reachability

Safety Games

- **Zero-sum Safety games** are **dual to reachability games**

Player A has a winning strategy in the safety game

$$\mathcal{G} = \langle V = V_A \uplus V_B, v_0, E, \Theta_A = S \rangle$$

iff

Player A has a winning strategy in the reachability game

$$\mathcal{G} = \langle V = V_A \uplus V_B, v_0, E, \Theta_B = V \setminus S \rangle$$

- In **concurrent multiplayer games**, we compute the winning region using the sets:

$$X_0^i := S$$

$$X_{n+1}^i := X_n^i \setminus \{v \mid \forall a_i \in \text{Act}_i, \text{Succ}(v, a_i) \cap (V \setminus X_n^i) \neq \emptyset\}$$

- From X_n^i , Player i has a strategy to stay in S at least for n rounds
- X_n^i decreases until reaching a fixed point $X_{n+1}^i = X_n^i$
- Then, The **winning region** of Player i is $W_i(\mathcal{G}) = X_n^i$ s.t. $X_{n+1}^i = X_n^i$

Safety Games

- **Zero-sum Safety games** are dual to reachability games

Player A has a winning strategy in the safety game

$$\mathcal{G} = \langle V = V_A \uplus V_B, v_0, E, \Theta_A = S \rangle$$

iff

Player A has a winning strategy in the reachability game

$$\mathcal{G} = \langle V = V_A \uplus V_B, v_0, E, \Theta_B = V \setminus S \rangle$$

- In **concurrent multiplayer games**, we compute the winning region using the sets:

$$X_0^i := S$$

$$X_{n+1}^i := X_n^i \setminus \{v \mid \forall a_i \in \text{Act}_i, \text{Succ}(v, a_i) \cap (V \setminus X_n^i) \neq \emptyset\}$$

- From X_n^i , Player i has a strategy to stay in S at least for n rounds
- X_n^i decreases until reaching a fixed point $X_{n+1}^i = X_n^i$
- Then, The **winning region** of Player i is $W_i(\mathcal{G}) = X_n^i$ s.t. $X_{n+1}^i = X_n^i$

ALTERNATIVE:

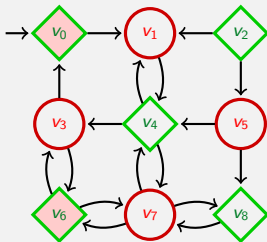
The opponents of Player i into a safety game has reachability objective $V \setminus S_i!$

- Consider the opponents of Player i as a single Player B that plays tuples of actions.
- Compute $\text{Attr}^B(V \setminus S_i)$
- Then, $W_i(\mathcal{G}) = V \setminus \text{Attr}^B(V \setminus S_i)$

Turn-based Zero-Sum Büchi Game

BÜCHI GAME:

$$\mathcal{G} = \langle V = V_A \uplus V_B, v_0, E, \text{BÜCHI}(T) \rangle \text{ for } T \subseteq V$$



• Winning condition for Player A :

$$\text{inf}(\pi) \cap T \neq \emptyset$$

• Solution complexity:

P_{TIME}

• Algorithm:

iterated attractors

• Dual game:

co-Büchi

Turn-based Zero-Sum Büchi Game : Attractors

- Let $X \subseteq V$.

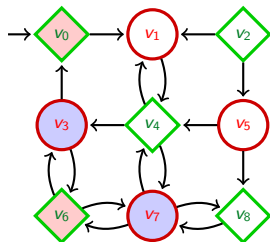
$$\text{Attr}_0^A(X) := \{v \in V_A \mid E(v) \cap X \neq \emptyset\} \\ \cup \{v \in V_B \mid E(v) \subseteq X\}$$

$$\text{Attr}_{n+1}^A(X) := \text{Attr}_n^A(X) \cup \\ \cup \{v \in V_A \mid E(v) \cap \text{Attr}_n^A(X) \neq \emptyset\} \\ \cup \{v \in V_B \mid E(v) \subseteq \text{Attr}_n^A(X)\}$$

- $\text{Attr}_n^A(X)$ increases until $\text{Attr}_{n+1}^A(X) = \text{Attr}_n^A(X)$
- Let $\text{Attr}^A(X)$ be the fixed point called **attractor in at least one step**
- Build $\text{Attr}^A(X)$ from which Player A has a strategy to reach X in **at least one move**

Turn-based Zero-sum Büchi Games : Attractor Example

Example



Player A : ○

Player B : ◇

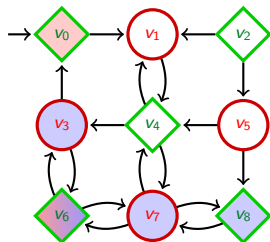
$T = \{v_0, v_6\}$

$$\text{Attr}_0^A(\{v_0, v_6\}) = \{v_3, v_7\}$$

Note the difference from Reachability!!!

Turn-based Zero-sum Büchi Games : Attractor Example

Example



Player A : ○

Player B : ◇

$T = \{v_0, v_6\}$

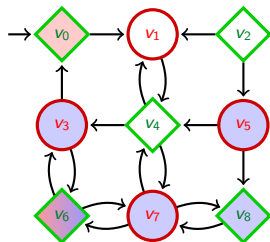
$$\text{Attr}_0^A(\{v_0, v_6\}) = \{v_3, v_7\}$$

Note the difference from Reachability!!!

$$\text{Attr}_1^A(\{v_0, v_6\}) = \{v_3, v_7, v_6, v_8\}$$

Turn-based Zero-sum Büchi Games : Attractor Example

Example



Player A : ○

Player B : ◇

$T = \{v_0, v_6\}$

$$\text{Attr}_0^A(\{v_0, v_6\}) = \{v_3, v_7\}$$

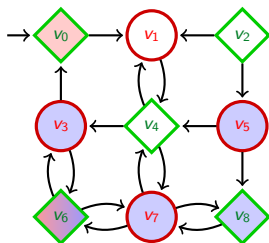
Note the difference from Reachability!!!

$$\text{Attr}_1^A(\{v_0, v_6\}) = \{v_3, v_7, v_6, v_8\}$$

$$\text{Attr}_2^A(\{v_0, v_6\}) = \{v_3, v_7, v_6, v_8, v_5\}$$

Turn-based Zero-sum Büchi Games : Attractor Example

Example



Player A : ○

Player B : ◇

$T = \{v_0, v_6\}$

$$\text{Attr}_0^A(\{v_0, v_6\}) = \{v_3, v_7\}$$

Note the difference from Reachability!!!

$$\text{Attr}_1^A(\{v_0, v_6\}) = \{v_3, v_7, v_6, v_8\}$$

$$\text{Attr}_2^A(\{v_0, v_6\}) = \{v_3, v_7, v_6, v_8, v_5\}$$

$$\text{Attr}_3^A(\{v_0, v_6\}) = \{v_3, v_7, v_6, v_8, v_5\}$$

Turn-based Zero-sum Büchi Games : Iterated Attractor

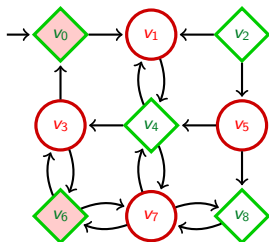
$$Z_n = \begin{cases} T & \text{if } n = 1 \\ \text{Attr}^A(Z_{n-1}) \cap T & \text{if } n > 1 \end{cases}$$

- Z_n is the set of final states from which **Player A** can force at least n visits of T
- The sequence $Z_1, Z_2, Z_3 \dots$ is decreasing
 - Base case: $T = Z_1 \supseteq Z_2 = \text{Attr}^A(Z_1) \cap T$
 - Inductive step: use the monotonicity of $\text{Attr}^A()$
 $Z_n \supseteq Z_{n+1} \Rightarrow \text{Attr}^A(Z_n) \supseteq \text{Attr}^A(Z_{n+1})$
 $\Rightarrow Z_{n+1} = \text{Attr}^A(Z_n) \cap T \supseteq \text{Attr}^A(Z_{n+1}) \cap T = Z_{n+2}$
- Let Z_∞ be the limit of the sequence $Z_1, Z_2, Z_3 \dots$:

$$Z_\infty = \bigcap_{n \geq 1} Z_n$$

Turn-based Zero-sum Büchi Games : Iterated Attractor Example

Example



$$Z_1 = \{v_0, v_6\}$$

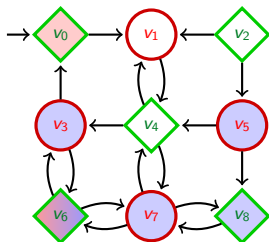
Player A : ○

Player B : ◇

$T = \{v_0, v_6\}$

Turn-based Zero-sum Büchi Games : Iterated Attractor Example

Example



Player A : ○

Player B : ◇

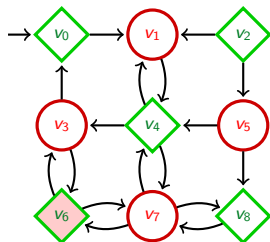
$T = \{v_0, v_6\}$

$$Z_1 = \{v_0, v_6\}$$

$$\text{Attr}^A(Z_1) = \{v_3, v_7, v_6, v_8, v_5\}$$

Turn-based Zero-sum Büchi Games : Iterated Attractor Example

Example



Player A : ○

Player B : ◇

$T = \{v_0, v_6\}$

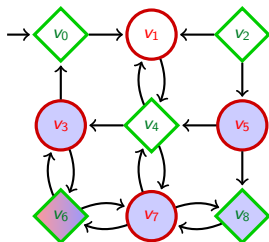
$$Z_1 = \{v_0, v_6\}$$

$$\text{Attr}^A(Z_1) = \{v_3, v_7, v_6, v_8, v_5\}$$

$$Z_2 = \{v_6\}$$

Turn-based Zero-sum Büchi Games : Iterated Attractor Example

Example



Player A : ○

Player B : ◇

$T = \{v_0, v_6\}$

$$Z_1 = \{v_0, v_6\}$$

$$\text{Attr}^A(Z_1) = \{v_3, v_7, v_6, v_8, v_5\}$$

$$Z_2 = \{v_6\}$$

$$\text{Attr}^A(Z_2) = \{v_3, v_7, v_6, v_8, v_5\}$$

Turn-based Zero-sum Büchi Games : Winning Regions and Strategies

Theorem

In a Büchi game \mathcal{G} where Player A targets $T \subseteq V$ infinitely often, the winning region of Player A is

$$W_A(\mathcal{G}) = \text{Attr}^A(Z_\infty)$$

Proof.

From $\text{Attr}^A(Z_\infty) \setminus Z_\infty$, play σ_A^1 to reach Z_∞ (similar to Reachability)

Once in $Z_\infty = \text{Attr}^A(Z_\infty) \cap F$

\Rightarrow Player A has a strategy σ_A^2 from Z_∞ to go back to Z_∞ in at least one step □

Turn-based Zero-sum Büchi Games : Winning Regions and Strategies

Theorem

In a Büchi game \mathcal{G} where Player A targets $T \subseteq V$ infinitely often, the winning region of Player B is

$$W_B(\mathcal{G}) = V \setminus \text{Attr}^A(Z_\infty)$$

Proof.

Let $v \in V \setminus \text{Attr}^A(Z_\infty) \Rightarrow \exists Z_n$ s.t. $v \in \text{Attr}^A(Z_n) \setminus \text{Attr}^A(Z_{n+1})$.

Put $rk(v) = n$ ($rk(v) = -1$ if $v \notin Z_1$)

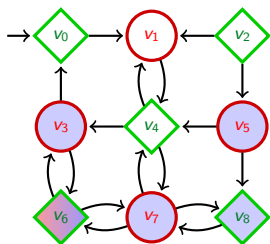
- If $v \in V_B$, $v \in V \setminus \text{Attr}^A(Z_{n+1}) \Rightarrow \exists v' \in E(v)$ s.t. $v' \notin Z_{n+1} = \text{Attr}^A(Z_n) \cap T$
 - if $v' \notin \text{Attr}^A(Z_n)$ but $v' \in T \Rightarrow rk(v') < n$ with $n \geq 1$ **Player B plays v' !**
 - else $v' \notin T$
- if $v \in V_A$, $v \in V \setminus \text{Attr}^A(Z_{n+1}) \Rightarrow \forall v' \in E(v)$ s.t. $v' \notin Z_{n+1} = \text{Attr}^A(Z_n) \cap T$
Player A cannot avoid v' s.t. $rk(v') < n$ or $v' \notin T$

In In the resulting play, we cannot have infinitely often $v' \in E(v)$ s.t. $rk(v') < rk(v)$

There is a position m s.t. $\forall j \geq m, v_j \notin T \Rightarrow$ Player B wins

Turn-based Zero-sum Büchi Games : Winning Regions and Strategies - Example

Example



Player A : ○

Player B : ◇

$T = \{v_0, v_6\}$

$$Z_\infty = \{v_6\}$$

$$W_A = \text{Attr}^A(Z_\infty) = \{v_3, v_7, v_6, v_8, v_5\}$$

$$W_B = V \setminus W_A = \{v_0, v_1, v_2, v_4\}$$

Concurrent Multiplayer Büchi Games

The Game: $\mathcal{G} = \langle \mathcal{AP}, \text{Ag}, (\text{Act}_i)_{i \in \text{Ag}}, V, v_0, \tau, E, (\text{BÜCHI}(T_i))_{i \in \text{Ag}} \rangle$

Decision Problem: Given the concurrent game \mathcal{G} , does Player i have a winning strategy?

- Similar algorithm as in the turn-based zero-sum setting
- Player i is the protagonist (Player A in the turn-based setting)
- The other players form the antagonist (Player B)
- Adapt the Attractor formula:

$$\text{Attr}_0^i(X) := \{v \in V \mid \exists a_i \in \text{Act}_i \text{ s.t. } \text{Succ}(v, a_i) \subseteq X\}$$

$$\text{Attr}_{n+1}^i(X) := \text{Attr}_n^A(X) \cup \{v \in V \mid \exists a_i \in \text{Act}_i \text{ s.t. } \text{Succ}(v, a_i) \subseteq \text{Attr}_n^i(X)\}$$

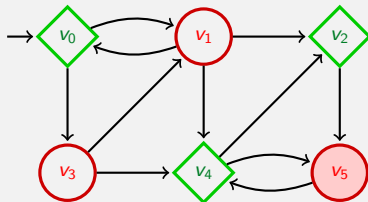
- Then, the sequence Z_n^i is defined by

$$Z_n^i = \begin{cases} T_i & \text{if } n = 1 \\ \text{Attr}^i(Z_{n-1}^i) \cap T_i & \text{if } n > 1 \end{cases}$$

Turn-based Zero-Sum co-Büchi Game

CO-BÜCHI GAME:

$$\mathcal{G} = \langle V = V_A \uplus V_B, v_0, E, \text{CO-BÜCHI}(T) \rangle \text{ for } T \subseteq V$$



- Winning condition for Player A :

$$\text{inf}(\pi) \cap T = \emptyset$$

- Solution complexity:

P_{TIME}

- Algorithm:

dualize + iterated attractors

- Dual game:

Büchi

- In **Zero-sum co-Büchi** games are **dual to Büchi games**

Player A has a winning strategy in the co-Büchi game

$$\mathcal{G} = \langle V = V_A \uplus V_B, v_0, E, \Theta_A = \text{CO-BÜCHI}(T) \rangle$$

iff

Player A has a winning strategy in the Büchi game

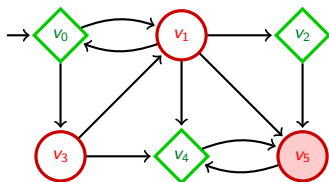
$$\mathcal{G} = \langle V = V_A \uplus V_B, v_0, E, \Theta_B = \text{BÜCHI}(T) \rangle$$

Player B is the protagonist!

- In **concurrent multiplayer game**: Use duality
 - The opponents of Player i have the objective $\text{BÜCHI}(T_i)$
 - Consider the opponents as a single player A that plays tuples of actions
 - Compute the winning region W_A of Player A in the Büchi game
 - Then, $W_i(\mathcal{G}) = V \setminus W_A$

co-Büchi Games : Winning Regions and Strategies - Example

Example



Player A : ○

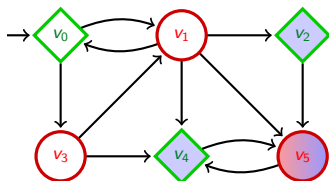
Player B : ◇

co-Büchi : $T = \{v_5\}$

- Player B (◇) has Büchi objective $T = \{v_5\}$

co-Büchi Games : Winning Regions and Strategies - Example

Example



Player A : ○

Player B : ◇

co-Büchi : $T = \{v_5\}$

- Player B (◇) has Büchi objective $T = \{v_5\}$
- Solve the Büchi game with B protagonist:

$$Z_1 = \{v_5\}$$

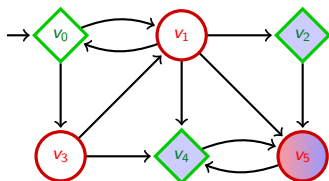
$$\text{Attr}^B(Z_1) = \{v_2, v_4, v_5\}$$

$$Z_2 = \{v_5\} = Z_\infty$$

$$W_B = \text{Attr}^B(Z_1) = \{v_2, v_4, v_5\}$$

co-Büchi Games : Winning Regions and Strategies - Example

Example



Player A : ○

Player B : ◇

co-Büchi : $T = \{v_5\}$

- Player B (◇) has Büchi objective $T = \{v_5\}$
- Solve the Büchi game with B protagonist:

$$Z_1 = \{v_5\}$$

$$\text{Attr}^B(Z_1) = \{v_2, v_4, v_5\}$$

$$Z_2 = \{v_5\} = Z_\infty$$

$$W_B = \text{Attr}^B(Z_1) = \{v_2, v_4, v_5\}$$

- In co-Büchi game: $W_A = V \setminus W_B = \{v_0, v_1, v_3\}$

- <http://www.lsv.ens-cachan.fr/~dwb/gtc.pdf>
- <https://www.irif.fr/~serre/Files/MPRI-notes-2015.pdf>
- <https://www.react.uni-saarland.de/teaching/automata-games-verification-12/downloads/notes10.pdf>
- Erich Grädel et al: *Automata, Logics, and Infinite Games - A Guide to Current Research*(available online)