Probabilities and Statistics - Lecture 5

Olariu E. Florentin

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Negative binomial distribution $NB(r,p)$

- Consider again a random experience which has only two outcomes: success or failure. A succes occurs with known probability p.
- We independently repeat this experience until r $(\in \mathbb{N}^*)$ successes occur.
- Let X be a variable counting the number of failures until we get r successes. This variable follows a negative binomial distribution with parameters r and p: $NB(r,p)$.
- Its distribution is

$$\begin{pmatrix} 0 & 1 & \dots & k & \dots \\ \binom{r-1}{r-1} p^r & \binom{r}{r-1} p^r(1-p) & \dots & \binom{k+r-1}{r-1} p^r(1-p)^k & \dots \end{pmatrix}$$

- Its characteristics are

$$\mathbb{E}[X] = \frac{r(1-p)}{p}, Var[X] = \frac{r(1-p)}{p^2}.$$

Negative binomial distribution $NB(r,p)$ - Example

Example. A medical researcher wants to recruit exactly 20 subjects for a study on an experimental drug for COVID-19. Each person that he (independently) interviews has a 60% chance of being eligible to participate in this study.

(a) Find the probability that he will have to refuse more than 10 people?

(b) What is the expected number of persons he has to refuse? But the expected number of inteviewed people?

Solution: (a) The number of persons he has to refuze is $X : NB(20, 0.6)$.

$$P(X > 10) = \sum_{k>10} \binom{k+19}{19} (0.6)^{20} (0.4)^k =$$

$$= 1 - \sum_{k\leqslant 10} \binom{k+19}{19} (0.6)^{20} (0.4)^k$$

(b) $\mathbb{E}[X] = 20 \cdot 0.4/0.6 \cong 13.33$. The expected number of inteviewed people is $\mathbb{E}[X] + 20 = 33.3$.

Hypergeometric distribution

- We recall the framework of the hypergeometric schema: in a box we have n balls of two different colors (n_1 white and n_2 black) we withdraw from the box, without replacement, k balls. Let X be the number of white balls we get; we say that this variable is hypergeometrically distributed.
- Its distribution is

$$X : \begin{pmatrix} 0 & 1 & \dots & j & \dots & r \\ \frac{\binom{n_1}{0}\binom{n_2}{k}}{\binom{n}{k}} & \frac{\binom{n_1}{1}\binom{n_2}{k-1}}{\binom{n}{k}} & \dots & \frac{\binom{n_1}{j}\binom{n_2}{k-j}}{\binom{n}{k}} & \dots & \frac{\binom{n_1}{r}\binom{n_2}{k-r}}{\binom{n}{k}} \end{pmatrix},$$

where $r = \min\{k, n_1\}$.
- Its characteristics are

$$\mathbb{E}[X] = \frac{kn_1}{n}, Var[X] = k \cdot \frac{n-k}{n-1} \cdot \frac{n_1}{n} \cdot \frac{n_2}{n}$$

Zipf distribution

- This is a empirical discrete distribution mainly used in linguistics although it was observed in many other areas.
- Zipf's law: the frequency of a value in a given range is almost equal with the inverse of its rank when sorted in decreasing order.
- Suppose that count the words of a book and arrange them in decreasing order of their frequency, then the frequency of a word is almost equal with the inverse of its rank in this order.
- For n objects, the distribution is

$$\begin{pmatrix} 1 & 2 & \dots & k & \dots & n \\ \frac{1}{H_n} \cdot \frac{1}{1} & \frac{1}{H_n} \cdot \frac{1}{2} & \dots & \frac{1}{H_n} \cdot \frac{1}{k} & \dots & \frac{1}{H_n} \cdot \frac{1}{n} \end{pmatrix}$$

- Its characteristics are $\mathbb{E}[X] = \frac{n}{H_n}$, $Var[X] = \frac{n^2}{H_n^2}$, where

$$H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}, (n \in \mathbb{N}^*).$$

Joint probability distributions

- Let X and Y two discrete random variables having distributions

$$X : \begin{pmatrix} x_1 & x_2 & \dots & x_n & \dots \\ p_1 & p_2 & \dots & p_n & \dots \end{pmatrix} \text{ and } Y : \begin{pmatrix} y_1 & y_2 & \dots & y_m & \dots \\ q_1 & q_2 & \dots & q_m & \dots \end{pmatrix}.$$

Definition 1

The joint probability distribution *of X and Y is the set*

$$(x_i, y_j, P\{X = x_i \cap Y = y_j\})_{i,j}$$

Joint probability distributions

- If we denote $r_{ij} = P\{X = x_i \cap Y = y_j\}$, the joint probability distribution of X and Y is

X across the columns, Y down the rows:

	x_1	x_2	$\cdots$	x_i	$\cdots$	
y_1	r_{11}	r_{21}	$\cdots$	r_{i1}	$\cdots$	q_1
y_2	r_{12}	r_{22}	$\cdots$	r_{i2}	$\cdots$	q_2
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	
y_j	r_{1j}	r_{2j}	$\cdots$	r_{ij}	$\cdots$	q_j
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	
	p_1	p_2	$\cdots$	p_i	$\cdots$	

- We note that the probabilities for the two variables can be obtained by adding up the values on rows (for Y) or on columns (for X):

$$\sum_i r_{ij} = q_j, \forall\, j \quad \text{and} \quad \sum_j r_{ij} = p_i, \forall\, i.$$

Joint probability distributions - example

Example. We have two boxes: B_1 which contains two white balls, two black and three red, and B_2 which contains three white balls, two black and one red. From the first box we withdraw a ball and we put it in the second; then, from the second box, we withdraw another ball. Let X be the number of white balls, and Y be the number of black balls obtained in the two withtdrawals.

(a) Determine the joint probability distribution of X and Y.

(b) Determine the distribution and the expectation of variable $X + Y$.

Solution: Obviously, X and Y are related in the following way: $X + Y \leqslant 2$. Let A_i be the event "the i-th ball is white", B_i be the event "the i-th ball is black", C_i be the event "the i-th ball is red" $(i = \overline{1,2})$.

Joint probability distributions - example

		X			
		0	1	2	
Y	0	6/49	11/49	8/49	25/49
	1	?	?	0	?
	2	?	0	0	?
		?	?	8/49	

Covariance of two random variables

Definition 2

Let X and Y two discrete random variables which have expectations.

(i) *The* covariance *of X and Y (if exists) is defined as*

$$cov[X, Y] = \mathbb{E}\left[(X - \mathbb{E}[X]) \cdot (Y - \mathbb{E}[Y])\right] =$$

$$= \sum_{i,j} (x_i - \mathbb{E}[X]) \cdot (y_j - \mathbb{E}[Y]) \cdot P\{X = x_i \cap Y = y_j\}.$$

(ii) *The* correlation coefficient *(or just the* correlation*) of X and Y (non-degenerate variables) is*

$$\rho(X, Y) = \frac{cov[X, Y]}{StDev[X] \cdot StDev[Y]}.$$

Joint probability distribution and covariance - example

Example. We have two boxes: B_1 containing 2 white balls, 2 black balls, and 3 red balls, and B_2 containing 3 white balls, 2 black balls, and 1 red ball. From B_1 we withdraw a ball which we put in B_2, then we withdraw another ball from the second box. Let X be the number of white withdrawn balls, and Y be the number of black withdrawn balls.

(a) Determine the joint probability distribution of X and Y.

(b) Determine the distribution and the expectation of XY.

(c) Determine the covariance and the correlation betwen X and Y.

Solution: We observe first that X and Y are related like this: $X + Y \leqslant 2$. Let A_i be the event "the i-th withdrawn ball is white", B_i be the event "the i-th withdrawn ball is black", and C_i be the event "the i-th withdrawn ball is red" $(i = \overline{1,2})$.

Joint probability distribution and covariance - example

		X = 0	1	2	
Y	0	6/49	11/49	8/49	25/49
	1	?	?	0	?
	2	?	0	0	?
		?	?	8/49	

Covariance of two random variables

Proposition 1

Let X and Y two random variables which have expectations and covariance. Then

(i) $cov[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$.

(ii) $Var[X + Y] = Var[X] + 2cov[X, Y] + Var[Y]$.

(iii) $-1 \leqslant \rho[X, Y] = \rho[Y, X] \leqslant 1$ *and* $\rho[X, X] = 1$ *(i. e.,* $cov[X, X] = Var[X]$*).*

(iv) *(exercise)* $\rho[aX + b, Y] = sgn(a) \cdot \rho[X, Y]$, *if* $a \in \mathbb{R}^*$, $b \in \mathbb{R}$.

(v) $cov[aX + bY + c, Z] = a \cdot cov[X, Z] + b \cdot cov[Y, Z]$, *for* $a, b, c \in \mathbb{R}$.

(vi) *(exercise)* $cov\left[\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right] = \sum_{i=1}^{n}\sum_{j=1}^{m} cov[X_i, Y_j]$.

Covariance of two random variables

proof: For (i)

$$cov[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] =$$

$$= \mathbb{E}[XY - \mathbb{E}[Y]X - \mathbb{E}[X]Y + \mathbb{E}[X]\mathbb{E}[Y]] =$$

$$= \mathbb{E}[XY] - 2\mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

For (ii)

$$Var[X + Y] = \mathbb{E}\left[(X + Y)^2\right] - \mathbb{E}^2[X + Y] =$$

$$= \mathbb{E}\left[X^2 + 2XY + Y^2\right] - \left(\mathbb{E}^2[X] + 2\mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}^2[Y]\right) =$$

$$= \left(\mathbb{E}\left[X^2\right] - \mathbb{E}^2[X]\right) + 2\left(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\right) + \left(\mathbb{E}\left[Y^2\right] - \mathbb{E}^2[Y]\right) =$$

$$= Var[X] + 2cov[X, Y] + Var[Y].$$

Covariance of two random variables

Now for (iii), since $0 \leqslant Var[tX+Y] = t^2 Var[X] + 2t \cdot cov[X,Y] + Var[Y]$, for every $t \in \mathbb{R}$, we must have:

$$\Delta = 4cov^2[X,Y] - 4Var[X]Var[Y] \leqslant 0 \Leftrightarrow$$

$$\Leftrightarrow |cov[X,Y]| \leqslant StDev[X] \cdot StDev[Y].$$

Then, $cov[X,X] = \frac{1}{2}(Var[2X] - 2Var[X]) = Var[X]$.

For (v):

$$cov[aX+bY+c, Z] = \mathbb{E}[aXZ + bYZ + cZ] - \mathbb{E}[aX+bY+c] \cdot \mathbb{E}[Z] =$$

$$= a\mathbb{E}[XZ] + b\mathbb{E}[YZ] + c\mathbb{E}[Z] - (a\mathbb{E}[X] + b\mathbb{E}[Y] + c) \cdot \mathbb{E}[Z]. \blacksquare$$

Covariance of two random variables - example

Example. Let X_1, Y_1 and X_2, Y_2 be two pairs of random variables with the following joint probability distributions:

		X_1		
		1	2	
Y_1	2	1/4	1/4	1/2
	4	1/4	1/4	1/2
		1/2	1/2	

		X_2		
		1	2	
Y_2	2	1/2	0	1/2
	4	0	1/2	1/2
		1/2	1/2	

Prove that $\rho[X_1, Y_1] \neq \rho[X_2, Y_2]$ and $cov[X_1, Y_1] \neq cov[X_2, Y_2]$.

Covariance of two random variables - example

Solution: X_1 and X_2 (Y_1 and Y_2) have the same distribution, $StDev[X_1] = StDev[X_2]$ and $StDev[Y_1] = StDev[Y_2]$, therefore it will be sufficient to show that: $cov[X_1, Y_1] \neq cov[X_2, Y_2]$.

$$\mathbb{E}[X_1] = \mathbb{E}[X_2] = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{3}{2}, \mathbb{E}[Y_1] = \mathbb{E}[Y_2] = 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} = 3.$$

We compute the expectation of $X_1 Y_1$ and the covariance of X_1 and Y_1:

$$X_1 Y_1 : \begin{pmatrix} 2 & 4 & 8 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \Rightarrow \mathbb{E}[X_1 Y_1] = 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{2} + 8 \cdot \frac{1}{4} = \frac{9}{2},$$

$$cov[X_1, Y_1] = \mathbb{E}[X_1 Y_1] - \mathbb{E}[X_1]\mathbb{E}[Y_1] = \frac{9}{2} - \frac{9}{2} = 0.$$

Covariance of two random variables - example

We now compute the expectation of $X_2 Y_2$ and the covariance of X_2 and Y_2

$$X_2 Y_2 : \begin{pmatrix} 2 & 8 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \Rightarrow \mathbb{E}[X_2 Y_2] = 2 \cdot \frac{1}{2} + 8 \cdot \frac{1}{2} = 5,$$

therefore

$$cov[X_2, Y_2] = \mathbb{E}[X_2 Y_2] - \mathbb{E}[X_2]\mathbb{E}[Y_2] = 5 - \frac{9}{2} = \frac{1}{2}. \clubsuit$$

Independent random variables

Definition 3

Two random variables X and Y are independent *if, for every $A \subseteq X(\Omega)$ and $B \subseteq Y(\Omega)$, we have*

$$P\{(X \in A) \cap (Y \in B)\} = P\{X \in A\} \cdot P\{Y \in B\}.$$

Since $P\{X = x_i \cap Y = y_j\} = P\{X = x_i\} \cdot P\{Y = y_j\} = p_i \cdot q_j$, it follows that the joint distribution can be computed like this: $r_{ij} = p_i q_j$.

Theorem 3.1

Let X and Y be two independent discrete random variables, then:

(i) $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

(ii) $Var[X+Y] = Var[X] + Var[Y]$.

(iii) $cov[X, Y] = 0$.

Independent random variables

proof: We consider only the case when both variables are finite. For (i):

$$\mathbb{E}[XY] = \sum_{z} zP\{XY = z\} = \sum_{z} z \cdot \left(\sum_{z=x_i y_j} P\{X = x_i \cap Y = y_j\} \right) =$$

$$= \sum_{i,j} x_i y_j P\{X = x_i \cap Y = y_j\} = \sum_{i,j} x_i y_j P\{X = x_i\} P\{Y = y_j\} =$$

$$= \left(\sum_{i} x_i P\{X = x_i\} \right) \cdot \left(\sum_{j} y_j P\{Y = y_j\} \right) = \mathbb{E}[X]\mathbb{E}[Y].$$

Independent random variables

From the last relation we get (iii): $cov[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0.$

For (ii) we use Proposition 1 or proceed directly:

$$Var[X+Y] = \mathbb{E}\left[(X+Y)^2\right] - (\mathbb{E}[X+Y])^2 =$$

$$= \mathbb{E}\left[X^2 + 2XY + Y^2\right] - (\mathbb{E}[X] + \mathbb{E}[Y])^2 =$$

$$= \mathbb{E}\left[X^2\right] + 2\mathbb{E}[XY] + \mathbb{E}\left[Y^2\right] - \mathbb{E}^2[X] - 2\mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}^2[Y] =$$

$$= \mathbb{E}\left[X^2\right] - \mathbb{E}^2[X] + \mathbb{E}\left[Y^2\right] - \mathbb{E}^2[Y] = Var[X] + Var[Y]. \blacksquare$$

The converse of this theorem is not necessarily true.

Independent random variables - example

Example. We roll two dice. Determine the expectation of the product and the variance of sum.

Solution: Let X_1 and X_2 the values on the dice. These two variables are independent (and have the same distribution), hence

$$\mathbb{E}[X_1X_2] = \mathbb{E}[X_1]\mathbb{E}[X_2] = \frac{49}{4} \text{ and}$$

$$Var[X_1 + X_2] = Var[X_1] + Var[X_2]. \clubsuit$$

Exercises for seminar

- Negative binomial distribution: I.1, I.3, I.4
- Joint probability distributions: II.2, II.4, II.7, II.8, II.11, II.15
- Reserve: I. 2, II.9, II.10, II.12

End

Exercises - Negative binomial distribution

I.1. We toss a die until we get four even faces.

(a) Find the probability that during these tosses six odd faces occur.

(b) What is the expected number of odd faces?

I.2. We withdraw cards (with replacement) from a regular deck of cards until we get five clubs.

(a) Find the probability that during these withdrawals we get five non-clubs cards.

(b) Find the expected number of non-clubs cards.

Exercises - Negative binomial distribution

I.3. We toss two dice until four doubles occur.

(a) Find the probability that during these tosses we get seven non-doubles.

(b) Find the expected number of non-doubles.

I.4. In a box we have three balls labeld with 0, two labeld with 1, and four labeled with 2. We withdraw two balls from the urn (with replacement) until we get three sums of 2.

(a) Compute the probability that during these withdrawals we get six sums different from 2.

(b) Compute the expected number of sums different from 2.

Exercises - Joint Probability Distributions

II.1. Show that (X, Y and Z are random variables)

(a) $cov[aX + bY + c, Z] = a \cdot cov[X, Z] + b \cdot cov[Y, Z], \forall a, b, c \in \mathbb{R}$.

(b) $cov\left[\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right] = \sum_{i=1}^{n}\sum_{j=1}^{m} cov[X_i, Y_j]$, for every random variables $(X_i)_{1\leqslant i\leqslant n}$ and $(Y_i)_{1\leqslant j\leqslant m}$ (use induction).

(c) $Var\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} Var[X_i] + 2\sum_{i<j} cov[X_i, X_j]$, for every random variables $X_1, X_2, \ldots, X_n$.

II.2. Let X be the random variable from below and $Y = X^2$. Show that X and Y are not independent but $cov[X, Y] = 0$.

$$X : \begin{pmatrix} -1 & 0 & 1 \\ 0.25 & 0.5 & 0.25 \end{pmatrix}$$

Exercises - Joint Probability Distributions

II.3. Suppose that X and Y have the following joint probability distribution

		Y		
		-3	2	4
X	1	0.2	?	0.2
	3	0.3	0.05	0.05

(a) Determine the distributions of X and Y.

(b) Compute $cov[X, Y]$ and $\rho[X, Y]$.

(c) Are X and Y independent?

Exercises - Joint Probability Distributions

II.4. A coin is flipped three times. Let X equals 1 if we get tail at the first flip and 0 otherwise, and Y a variable which equals the number of tails in all three flips. Find

(a) the JPD of the two variables.

(b) the distributions of X and Y and their covariance.

II.5. Let X be a random variable and $Y = X^2$. We know that

$$\begin{pmatrix} -2 & -1 & 1 & 2 \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{pmatrix}.$$

Find

(a) the distribution of Y and the JPD of X and Y.

(b) the covariance and the correlation of X and Y.

Exercises - Joint Probability Distributions

II.6. Let X and Y two random independent variables such that

$$X : \begin{pmatrix} 1 & -1 \\ 0.6 & 0.4 \end{pmatrix}, \; Y : \begin{pmatrix} -1 & 0 & 1 \\ 0.2 & 0.5 & 0.3 \end{pmatrix}.$$

(a) Determine their JPD and covariance.

(b) Determine the distribution and the expectation of $X + Y$.

II.7. In a box we have three red and five black balls. We withdraw a ball from the box and we replace it by one of opposite colour. Then we withdraw another ball from the box. Let X be the number of red and Y the number of black withdrawn balls.

(a) Determine the JPD of variables X and Y.

(b) Are X and Y independent?

Exercises - Joint Probability Distributions

II.8. In a box we have four withe balls (two labeled with 1 and two labeled with 2) and three black balls (two labeled with 1 and one labeled 2). We with draw, one by one without replacement, two balls. Let X be the number of white balls and Y the number of balls labeled with 2.

(a) Find the JPD of X and Y.

(b) Variables X and Y are independent?

II.9. A coin is tossed three times. Let X be the number of tails obtained at the first two tossings and Y be the number of tails at the last tossing. Determine

(a) the distribution of X and Y.

(b) the JPD of X and Y. (Are X and Y independent?)

(c) the distribution of $X + Y$.

Exercises - Joint Probability Distributions

II.10. X and Y have the following JPD

		X			
		-1	1	2	3
Y	-1	0	$1/36$	$1/6$	$1/12$
	0	$1/18$	0	$1/18$	0
	1	0	$1/36$	$1/6$	$1/12$
	2	$1/12$	0	$1/12$	?

(a) Compute $P(X \geqslant 2 \text{ and } Y \leqslant 0)$.

(b) Are X and Y independent?

(c) Determine the distribution of $X + Y$.

Exercises - Joint Probability Distributions

II.11. Two players P_1 and P_2 compete in a match of tennis. The winner is the first player to win two sets in a best-of-three. P_1 independently wins a set with probability $1/3$. Let X be the number of the sets played by P_1 up to the end of the match and by Y be the number of sets P_2 wins in this match. Determine

(a) the joint probability distribution of X and Y;

(b) the covariance of the two variables; X and Y are independent variables?

II.12. We have a biased coin: the probability of heads in a ny given toss is $1/3$. We toss the coin three times. Let X be the number of tail occurrences and Y be the maximum number of head occurrences in a row. Find

(a) the joint probability distribution of X and Y;

(b) the covariance of the two variables; X and Y are independent variables?

Exercises - Joint Probability Distributions

II.13. A die is rolled three times. X is a variable that denotes how many times we get an even number, and Y denotes how many times we get an prime number. Determine

(a) the joint probability distribution of X and Y;

(b) the covariance of the two variables; are X and Y independent?

II.14. A box contains 5 white and 4 red balls. We withdraw at random a ball from the box and we replaced it with one of opposite color. Then, we withdraw another ball from the box. Let X be the number of white balls and Y be the number of red balls obtained. Determine

(a) the joint probability distribution of X and Y; what is the functional relation between the two random variables?

(b) the covariance of the two variables; are X and Y independent?

Exercises - Joint Probability Distributions

II.15. A box contains 3 black and 5 green balls. We withdraw a ball from the box and, if we get a black one we return the ball in the box together with a green one, otherwise we replace it with two black balls. Then, we withdraw another ball from the box. Let X be the number of black balls and Y be the number of green balls obtained. Find

(a) the joint probability distribution of X and Y

(b) the covariance of X and Y; are X and Y independent?

II.16. We have two boxes: B_1 contains 2 white and 2 black balls, and B_2 contains 1 white and 2 black balls. We roll a die and, if we get a multiple of 3 we withdraw a ball from B_1, otherwise we withdraw a ball from B_2. Let X be the the number of white balls remaining in B_1 and Y be the number of black balls remaining in B_2.

(a) Find the joint probability distribution of X and Y.

(b) Find the covariance of X and Y; are X and Y independent?

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