Probabilities and Statistics - Lecture 4

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March 9, 2026

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Discrete random variables - Introduction

- After we perform a random experiment we are often interested in computing the value of (or to measure) a function which is associated with the possible outcome: sum/product of two dice, the number of heads when tossing a coin etc.
- This it is possible because in many cases the outcomes are quantitative ones (real or integer numbers).
- The numerical result of the measurement of an outcome it is called a *random variable* - because of its unknown variation.
- Informally a *random variable* is a function associating to each random event a number - which may be the result of a measurement.

Distribution of a discrete random variable

Definition 1

Let $\mathcal{E}$ a random experience, Ω the sample space, a real random variable *is a function $X : \Omega \to \mathbb{R}$, such that for every interval $J \subseteq \mathbb{R}$, $X^{-1}(J)$ is a random event.*

Definition 2

A random variable *is called* discrete*, if its image is a countably (finite or infinite) set, that is $|X(\Omega)| \leqslant \aleph_0$. Otherwise it is a* continuous random variable.

- If the sample space is a discrete set (i.e., $|\Omega| \leqslant \aleph_0$), then an associated random variable can only be discrete.

Distribution of a discrete random variable

- If $X(\Omega) = \{x_1, x_2, \ldots, x_n, \ldots\}$, then the set of all pairs (x_i, p_i) is the *distribution* or the *repartition* of the discrete random variable X, denoted by the following table

$$X : \begin{pmatrix} x_1 & x_2 & \ldots & x_n & \ldots \\ p_1 & p_2 & \ldots & p_n & \ldots \end{pmatrix} \quad (1)$$

- We use the notations $P\{X = x_i\} = P(X = x_i) = p_i$. Obviously, the sum (which may be a series) of the probabilities is 1:

$$\sum_i p_i = 1, 0 < p_i \leqslant 1, \forall\, i$$

Distribution of a discrete random variable

Definition 3

Let $X : \Omega \to \mathbb{R}$ *discrete random variable The* probability mass function *of* X *is* $f_X : X(\Omega) \to [0,1]$, *defined by* $f(x_i) = p_i = P\{X = x_i\}$, $\forall\, x_i \in X(\Omega)$.

- The above function completely defines a discrete random variable: the information related to the most refined partition (by X) of the sample space is contained by this function.
- It is the case that two different random variables may have the same distribution.
- A random variable X is often called simply a distribution or repartition, covering in this way the entire family of random variables which have the same probability mass function as X.

Distribution of a discrete random variable - example

Example. In a box we have 4 white, 3 red, and 3 blue balls. We withdraw, without replacement, 2 balls. For each white ball we win 1$ and for each blue ball we loose 1$. Let X be the win; determine the distribution of X.

Solution: The possible values of X are $\{\pm 1, 0, \pm 2\}$; we determine the probability mass function (using hypergeometric schema)

$$f_X(-2) = P\{X = -2\} = \frac{\binom{3}{2}}{\binom{10}{2}} = \frac{3}{45} = \frac{1}{15},$$

$$f_X(-1) = P\{X = -1\} = \frac{\binom{3}{1} \cdot \binom{3}{1}}{\binom{10}{2}} = \frac{9}{45} = \frac{3}{15},$$

Distribution of a discrete random variable - example

$$f_X(0) = P\{X = 0\} = \frac{\binom{3}{2} + \binom{4}{1} \cdot \binom{3}{1}}{\binom{10}{2}} = \frac{15}{45} = \frac{1}{3},$$

$$f_X(1) = P\{X = 1\} = \frac{\binom{4}{1} \cdot \binom{3}{1}}{\binom{10}{2}} = \frac{12}{45} = \frac{4}{15},$$

$$f_X(2) = P\{X = 2\} = \frac{\binom{4}{2}}{\binom{10}{2}} = \frac{6}{45} = \frac{2}{15}.$$

Distribution of a discrete random variable - example

The distribution of X is

$$X : \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ \frac{1}{15} & \frac{3}{15} & \frac{5}{15} & \frac{4}{15} & \frac{2}{15} \end{pmatrix}. \clubsuit$$

Expectation of a discrete random variable

Definition 4

Let X be a discrete random variable having a distribution as above, its expectation *(if exists) is*

$$\mathbb{E}[X] = \sum_i p_i x_i \tag{2}$$

- The expectation of a discrete random variable is a finite sum or a series (which may converge or not) of its values with corresponding probabilities as weights.
- If the right member contains a divergent series, then we say that X does not have expectation.

Expectation of a discrete random variable - Example

Example. We roll two dice and denote by X the maximum value. X is a discrete random variable with values: $1, 2, \ldots, 6$. Its probability mass function is

$$\begin{aligned}
f_X(1) = P\{X = 1\} = \; & P(\{(1,1)\}) = \frac{1}{36}, \\
f_X(2) = P\{X = 2\} = \; & P(\{(1,2),(2,1),(2,2)\}) = \frac{1}{12}, \\
f_X(3) = P\{X = 3\} = \; & P(\{(1,3),(2,3),(3,3),(3,2),(3,1)\}) = \frac{5}{36}, \\
f_X(4) = P\{X = 4\} = \; & P(\{(1,4),(2,4),(3,4),(4,4),(4,3),\ldots\}) = \frac{7}{36}, \\
f_X(5) = P\{X = 5\} = \; & P(\{(1,5),(2,5),(3,5),(4,5),(5,5),\ldots\}) = \frac{1}{4}, \\
f_X(6) = P\{X = 6\} = \; & P(\{(1,6),(2,6),(3,6),(4,6),(5,6),\ldots\}) = \frac{11}{36}.
\end{aligned}$$

Expectation of a discrete random variable - Example

Its repartition is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{36} & \frac{1}{12} & \frac{5}{36} & \frac{7}{36} & \frac{1}{4} & \frac{11}{36} \end{pmatrix}.$$

Its expectation is

$$\mathbb{E}[X] = 1 \cdot \frac{1}{36} + 2 \cdot \frac{1}{12} + 3 \cdot \frac{5}{36} + 4 \cdot \frac{7}{36} + 5 \cdot \frac{1}{4} + 6 \cdot \frac{11}{36} = \frac{161}{36}. \clubsuit$$

Proposition 1

Let X be a discrete random variable and let $h : \mathbb{R} \to \mathbb{R}$ be a real function. Then $h(X)$ is a discrete random variable and

$$\mathbb{E}[h(X)] = \sum_i h(x_i) p_i,$$

if this expectation exists.

Expectation of a discrete random variable

proof: Let $Y = h(X) : \Omega \to \mathbb{R}$; Y is a discrete random variable. Probability mass function of Y is $f_Y : Y(\Omega) = h(X(\Omega)) \to [0,1]$, where

$$f_Y(y_j) = P\left(\bigcup_i \{h(x_i) = y_j\}\right) = \sum_{h(x_i)=y_j} P\{X = x_i\} = \sum_{h(x_i)=y_j} f_X(x_i),$$

Hence

$$\mathbb{E}[Y] = \sum_j y_j f_Y(y_j) = \sum_j y_j \sum_{h(x_i)=y_j} f_X(x_i) =$$

$$= \sum_j \sum_{h(x_i)=y_j} h(x_i) f_X(x_i) = \sum_i h(x_i) f_X(x_i).$$

■

Expectation of a discrete random variable

Proposition 2

(i) *If X is a discrete random variable having expectation, then $aX+b$ is a discrete random variable and $\mathbb{E}[aX+b]=a\mathbb{E}[X]+b$, for every $a,b\in\mathbb{R}$.*

(ii) *If X_1 and X_2 are discrete random variables having expectation, then X_1+X_2 is a discrete random variable and $\mathbb{E}[X_1+X_2]=\mathbb{E}[X_1]+\mathbb{E}[X_2]$.*

(iii) *Let $X\geqslant 0$ be a discrete random variable having expectation, then $\mathbb{E}[X]\geqslant 0$, and $\mathbb{E}[X]=0$ if and only if $X\equiv 0$.*

Expectation of a discrete random variable

proof: (i) The proof is obvious using Proposition 1.

For (ii) We can write the expectation of X like follows

$$\mathbb{E}[X] = \sum_{\omega\in\Omega} X(\omega)P(\omega),$$

$$\mathbb{E}[X_1 + X_2] = \sum_{\omega\in\Omega} \left[X_1(\omega) + X_2(\omega)\right] P(\omega) =$$

$$= \sum_{\omega\in\Omega} X_1(\omega)P(\omega) + \sum_{\omega\in\Omega} X_2(\omega)P(\omega) = \mathbb{E}[X_1] + \mathbb{E}[X_2].$$

(iii) If $x_i \geqslant 0$, for every i, then $p_i x_i \geqslant 0$, $\forall i$ and

$$\mathbb{E}[X] = \sum_i p_i x_i \geqslant 0.$$

■

Expectation of a discrete random variable - Example

Example. We roll two dice. Compute the expectation of their sum.

Solution: Let X_i be the value on the i-th dice; the sum is $X = X_1 + X_2$, hence, using the above proposition, $\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2]$. Variables X_1 and X_2 have the same distribution:

$$X_1, X_2 : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}.$$

$$\mathbb{E}[X_1] = \mathbb{E}[X_2] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{21}{6} = \frac{7}{2}$$

$$\Rightarrow \mathbb{E}[X] = 7. \clubsuit$$

Variance of a discrete random variable

Definition 5

Let X be a discrete random variable. Its variance *(if exists) is:*

$$Var[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_i p_i \left(x_i - \mathbb{E}[X]\right)^2 .$$

- A necessary condition for the existence of variance is the existence of its expectation. A variance computing rule follows.

Proposition 3

Let X be a random variable which has variance, then

$$Var[X] = \mathbb{E}[X^2] - \left(\mathbb{E}[X]\right)^2 ,$$

provided that the involved expectations exist.

Variance of a discrete random variable

proof:

$$Var[X] = \sum_i p_i(x_i - \mathbb{E}[X])^2 = \sum_i p_i \left(x_i^2 - 2x_i\mathbb{E}[X] + (\mathbb{E}[X])^2\right) =$$

$$= \sum_i p_i x_i^2 - 2\left(\sum_i p_i x_i\right)\mathbb{E}[X] + \sum_i p_i(\mathbb{E}[X])^2 =$$

$$= \mathbb{E}[X^2] - 2\mathbb{E}[X]\cdot\mathbb{E}[X] + (\mathbb{E}[X])^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 .$$

■

Proposition 4

Let X be a random variable which has variance, then

(i) $Var[X] \geqslant 0$; $Var[X] = 0$ *if and only if* $X \equiv const$ *(that is, X is degenerate random variable);*

(ii) $Var[aX + b] = a^2 Var[X]$, *for every* $a, b \in \mathbb{R}$.

Variance of a discrete random variable

proof: (i) obviously, $Var[X] \geqslant 0$ and $Var[X] = 0$ if and only if $x_i = \mathbb{E}[X]$, $\forall i$, i.e. X is a constant (equal with its expectation).
(ii) $Var[aX + b] = \mathbb{E}[(aX + b - a\mathbb{E}[X] - b)^2] = \mathbb{E}[a^2(X - \mathbb{E}[X])^2] =$
$= a^2\mathbb{E}[(X - \mathbb{E}[X])^2]$. ■

Definition 6

The standard deviation *of a random variable X which has variance is*

$$StDev[X] = \sqrt{Var[X]}.$$

Uniform distribution U_n

- A random variable is uniformly distributed *with parameter* $n \in \mathbb{N}^*$ if it has the following distribution

$$U_n : \begin{pmatrix} 1 & 2 & \dots & n \\ 1/n & 1/n & \dots & 1/n \end{pmatrix}$$

- It's easy to see that, if $X \sim U_n$, then

$$\mathbb{E}[X] = \frac{n+1}{2} \text{ and } Var[X] = \frac{n^2-1}{12}.$$

- We already used this distribution in the case of rolling a die.

Bernoulli and Binomial $B(n,p)$ distributions

- Consider a random experience which has only two outcomes: success or failure. Let X be defined by

$$X = \begin{cases} 1, & \text{for success} \\ 0, & \text{for failure.} \end{cases}$$

(Usually we associate a random event, A, with this experience ($P(A) = p$): the success means the occurrence of this event.)

- Probability mass function of X is $f(0) = 1 - p$ and $f(1) = p$. We say that such a variable has a Bernoulli distribution. If $X \sim B(p)$, then

$$X : \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}.$$

- Its expectation and variance are

$$\mathbb{E}[X] = p, Var[X] = p(1-p).$$

Bernoulli and Binomial $B(n, p)$ distributions

- Suppose now that such an experiment is independently performed n times and let X be the number of successes.
- We say that X is binomially distributed with *parameters n and p*. Using the binomial schema we can find its distribution, $B(n,p)$:

$$\begin{pmatrix} 0 & 1 & \dots & k & \dots & n \\ (1-p)^n\binom{n}{0} & p(1-p)^{n-1}\binom{n}{1} & \dots & p^k(1-p)^{n-k}\binom{n}{k} & \dots & p^n\binom{n}{n} \end{pmatrix},$$

- If $X \sim B(n,p)$, then $\mathbb{E}[X] = np$ and $Var[X] = np(1-p)$.

Proposition 5

Let $X_1, X_2, \dots, X_n$ be independent Bernoulli distributed random variables with parameter $p \in (0,1)$. Then $X = \sum_{i=1}^{n} X_i$ is a binomial variable distributed $B(n,p)$.

Binomial $B(n,p)$ distribution - Examples

Example.

(a) We roll a die until 6 occurs three times. What is the probability that exactly twenty rolls are sufficient?

(b) If the die is twenty times rolled what is the expected number of occurrences of value 6?

Solution: (a) Exactly twenty rolls are enough if and only if 6 occurs two times in the first nineteen rolls and occurs one more at the last roll. These two events are independent, hence the required probability is

$$\binom{19}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{17}\cdot\frac{1}{6}\cong 0.035682$$

(b) Variable X which numbers the occurences of 6 is $B(20,1/6)$ distributed. $\mathbb{E}[X]=20\cdot\frac{1}{6}=\frac{10}{3}\cong 3.333$ ♣

Binomial $B(n, p)$ distribution - Examples

Example. O variant of the "Wheel of fortune" is the following: a player bets on a number from 1 to 6, then rolls three dices and, if its number occurs k times, the palyer wins $k\$$ $(1 \leqslant k \leqslant 3)$, otherwise he looses 1\$. Is this a fair game? What is the expected win?

Solution: Let X be the amount the player wins; the possible values of X are $\{-1, 1, 2, 3\}$. X has a probability mass function very similar to that of a binomial distribution. We have

$$P\{X = -1\} = \binom{3}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^3 = \frac{125}{216},$$

$$P\{X = 1\} = \binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2 = \frac{75}{216},$$

Binomial $B(n, p)$ distribution - Examples

$$P\{X = 2\} = \binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^1 = \frac{15}{216},$$

$$P\{X = 3\} = \binom{3}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^0 = \frac{1}{216}.$$

The game is not a fair one as the chances of loosing are $125/216 > 1/2$; the expected win (another way to measure the fairness of the game) is

$$\mathbb{E}[X] = (-1)\cdot\frac{125}{216} + 1\cdot\frac{75}{216} + 2\cdot\frac{15}{216} + 3\cdot\frac{1}{216} = -\frac{17}{216}. \clubsuit$$

Geometric distribution *Geometric*(p)

- Consider now a random experience and a related random event A (with $P(A) = p \in (0,1)$). The variable which gives the number of independent performings of the experiment until event A occurs is geometrically distributed *with parameter p*.
- The distribution of such a variable (see the geometric schema) is

$$G(p) : \begin{pmatrix} 1 & 2 & \dots & n & \dots \\ p & p(1-p) & \dots & p(1-p)^{n-1} & \dots \end{pmatrix}$$

- Its characteristics are

$$\mathbb{E}[X] = \frac{1}{p} \text{ and } Var[X] = \frac{1-p}{p^2}.$$

Geometric distribution $Geometric(p)$ - Example

Example. We repeatedly roll two dice until we get a product equal with 6. What is the expectation and the variance of the number of rolls?
Solution: $A =$"the product of the two values is 6"

$$A = \{(1,6),(2,3),(3,2),(6,1)\}, P(A) = \frac{4}{36}.$$

Let $X =$ the number of rolls until A occurs. X is geometrically distributed with parameter $p = 1/9$.

$$\mathbb{E}[X] = \frac{1}{p} = 9, Var[X] = \frac{1-p}{p^2} = \frac{8}{9}.$$

We expect 9 rolls in order to get product of 6. ♣

Poisson distribution, $Poisson(\lambda)$

- A random variable X is Poisson distributed *with cu parameter* $\lambda > 0$ if its probability mass function is

$$f(k) = P\{X = k\} = e^{-\lambda}\frac{\lambda^k}{k!}, \forall k \geqslant 0.$$

- Its distribution is

$$Poisson(\lambda) : \begin{pmatrix} 0 & 1 & \dots & n & \dots \\ \lambda^0 \frac{e^{-\lambda}}{0!} & \lambda^1 \frac{e^{-\lambda}}{1!} & \dots & \lambda^n \frac{e^{-\lambda}}{n!} & \dots \end{pmatrix},$$

- Its characteristics are

$$\mathbb{E}[X] = \lambda \text{ and } Var[X] = \lambda.$$

Poisson distribution, $Poisson(\lambda)$

- Poisson distribution approximate the number of so called "accidents" - events that occur with low frequency in a given interval (say, of time, or space), if these events occur with a known average rate.
- Examples of Poisson distribution are: the number of wrong phone calls received per hour, the number of decay events per second from a radioactive source, the number of printing errors on a page, the number of births in a day etc.
- Such applications are in part due to the fact that for a sufficiently big n and a very small p (such that np has a reasonable value) $B(n,p)$ can be approximated with $Poisson(np)$.

Poisson distribution, $Poisson(\lambda)$ - Example

Example. In a hospital, the births have a rate of 2.1 per hour.

(a) What is the probability that, in a given hour, we have four births?

(b) What is the probability that, in a given hour, we have at least three births?

Solution: Let X be the number of births in the given hour, $X \sim Poisson(2.1)$.

(a) $P\{X = 4\} = \lambda^4 \dfrac{e^{-\lambda}}{4!} \cong 0.099231\ (\lambda = 2.1)$

(b) For the second requirement

$$P\{X \geqslant 3\} = \sum_{k \geqslant 3} P\{X = k\} = 1 - \sum_{k=0}^{2} P\{X = k\} =$$

$$= 1 - P\{X = 0\} - P\{X = 1\} - P\{X = 2\} =$$

$$= 1 - e^{-\lambda} - \lambda e^{-\lambda} - \frac{\lambda^2}{2} e^{-\lambda} \cong 0.350368\ (\lambda = 2.1). \clubsuit$$

Exercises for seminar

- Discrete random variables I.1, I.2, I.3, I.6, I.9, I.11. I.15.
- Remarkable discrete distribution: II.1, II.5, II.7, II.4, II.7, III.1, III.2, III.4, IV.1, IV.2, IV.4.
- Reserve: I.7, I.8, II.3, III.3, IV.5

End

Exercises - Discrete random variables

I.1. Two balls are randomly chosen from a bag which contains 8 white balls, 4 black, and 2 yellow. Suppose that a black ball worth 2$, and a white one 1$. Let X be the total win. Determine the distribution of X and compute its expectation and its variance.

I.2. A biased coin has the probability of tail occurrence $2/3$. We flip the coin four times. Let X be the maximum numbers of tails in the row. Compute the distribution, the expectation, and the variance of X.

I.3. Let X be the difference between the number of head occurrences and the number of tail occurrences in three flips of a fair coin. Compute the distribution, the expectation, and the variance of X.

Exercises - Discrete random variables

I.4.

(a) We roll a die and denote by X the value. Determine the distribution, the expectation, and the variance of X.

(b) We roll two dice. What are the expectation and the variance of the sum? Same question for the product.

I.5. We roll a die twice. Let X_1 and X_2 be the two values. Define $X = \min\{X_1, X_2\}$ and $Y = \max\{X_1 + X_2, X_1 \cdot X_2\}$. Determine the distributions of a) X and b) Y.

Exercises - Discrete random variables

I.6. We have three urns. The first contains one white ball and one black ball, the second, two white and six black balls, and the third, one white and three black balls. From the first urn we withdraw a ball and introduce it in the second; from the second we withdraw a ball and introduce it in the third; at last we withdraw a ball from the third urn. Compute the expectation and the variance of the number of white withdrawn balls.

I.7. A royal family is determined to have another child as long as they don't have a boy or there are at most three children in their family. Compute the expectation and the variance of the number of girls in such a family. (The chances that a new born is a girl are 1/2.)

Exercises - Discrete random variables

I.8. A coin is flipped until we get four tails or four heads (whichever comes first). Compute the expectation and the variance of the number of required flips.

I.9. We have three boxes. First box contains two white and two black balls, the second contains five white and three black balls, and the third contains three white and three black balls. We withdraw a ball from each box. Determine the distributions, the expectations, and the variance of the number of black withdrawn balls.

I.10*. Five numbers are randomly and uniformly distributed to five players labeled from 1 to 5. When two players faces their numbers the greater number wins. First, players 1 and 2 compare their numbers, then the winner plays with player 3, and so on. Let X the number of wins of player 1. Determine the distribution and the expectation of X.

Exercises - Discrete random variables

I.11. Two players P_1 and P_2 compete in a match of tennis. The winner is the first player to win two sets in a best-of-three. P_1 independently wins a set with probability $1/3$. Let X be the number of the sets played by P_1 up to the end of the match and by Y be the number of sets P_2 wins in this match. Determine the distributions and the expectations of X and Y.

I.12. We have a biased coin: the probability of a head in any given toss is $1/3$. We toss the coin three times. Let X be the number of tail occurrences and Y be the maximum number of head occurrences in a row. Find the distributions and the expectations of X and Y.

I.13. A die is rolled three times. X is a variable that denotes how many times we get an even number, and Y denotes how many times we get a prime number. Determine the distributions and the expectations of X and Y.

Exercises - Discrete random variables

I.14. A box contains 5 white and 4 red balls. We withdraw at random a ball from the box and we replaced it with one of opposite color. Then, we withdraw another ball. Let X be the number of white balls and Y be the number of red balls obtained. Find the repartitions of X and Y.

I.15. A box contains 3 black and 5 green balls. We withdraw a ball from the box; if we get a black one we return it in the box together with a green one, otherwise we replace it with two black balls. Then, we withdraw another ball from the box. Let X be the number of black and Y be the number of green balls obtained. Find the distributions of X and Y.

I.16. We have two boxes: B_1 contains 2 white and 2 black balls, and B_2 contains 1 white and 2 black balls. We roll a die and, if we get a multiple of 3 we withdraw a ball from B_1, otherwise we withdraw a ball from B_2. Let X be the the number of white balls remaining in B_1 and Y be the number of black balls remaining in B_2. Find the distributions

Exercises - Discrete random variables

I.17. Let X a random variable which has expectation and variance ($\mathbb{E}[X] = \mu$ and $Var[X] = \sigma^2 > 0$). Compute the expectation and the variance of the variable $\frac{X - \mu}{\sigma}$ (this operation is called *standardization*).

I.18. Prove that $Var[X + Y] + Var[X - Y] = 2\,Var[X] + 2\,Var[Y]$.

I.19. Let X and Y be two independent random variables having the same expectation and the same variance. Show that $M\left[(X - Y)^2\right] = 2\,Var[X]$.

I.20. If X and Y have the same variance, then

$$\mathbb{E}[(X + Y)(X - Y)] = \mathbb{E}[X + Y]\mathbb{E}[X - Y].$$

Exercises - Binomial distribution

II.1. We flip two coins seven times. What is the expected number of pair of heads (head on both coins)?

II.2. A source randomly (and independently) generates bits (0 with probability 0.6).

(a) What is the probability that in sequence of seven consecutive bits we get five 0's and two 1's?

(b) What is the expected number of 0's in a sequence of five consecutive bits?

II.3. At the begining of XX century the rate of succes for contacting by telephone a certain person was 0.75. What was the expected number of successes in twelve such attempts?

Exercises - Binomial distribution

II.4. A certain person claims the it has extra-senzorial perception (ESP); He is tested like follows: a coin is ten times flipped and he is asked to guess the results. Seven answers are correct. What is the probability the a normal person gets a result as good as this person?

II.5. We withdraw ten cards from a standard deck, every time returning the chosen card. What is the expected number of clubs obtained?

II.6. A communication channel randomly and independently transmits bits (0 with probability 0.4). There are received eight pairs. What is the expectation and the variance of the number of received $0-1$ pairs?

II.7. We roll two dice eight times. What is the expected numer of rolls which gives an even product?

Exercises - Geometric distribution

III.1. What is the expected number of rolls of two dice in order to get a product less than 7? Same question for an even sum.

III.2. We pick cards from a standard deck - every time returning the withdrawn card. What is the expected number of withdrawals in order to get a club?

III.3. 5% of transmitted bits through a certain communication channel are erroneously recepted. The bits are received until the first error. What is the expected number of received bits?

III.4. We withdraw a card from a standard deck (every time returning it in the deck). What is the expected number of withdrawals in order to get a card which is not a diamond?

III.5. We roll two dice many times. What is the expected number of rolls until we get a product that is a prime number? What is the expected number of rolls until we get a sum which is a multiple of 5?

Exercises - Poisson distribution

IV.1. During the interval 7:00 - 8:00 the expected number of highway accidents is 0.7. What is the probability that in this interval

a) there are at least three accidents?

b) there is at least one accident?

IV.2. A transport company has three cars for rental. The demand is a Poisson variable with parameter $\lambda = 1.5$. Compute the proportion of days in which

(a) no car is demanded;

(b) the company has a full book.

IV.3. Suppose that the number of typographical errors follows Poisson distribution with a mean of 3 errors per page. Find the probability that on a given page we have at least 4 such errors.

Exercises - Poisson distribution

IV.4. The expected number of plane landings per minute in a certain airport is 3. Determine the probability that in a given minute land at most 2 airplanes.

IV.5. The expected number of requests per minute for a webserver is 4. Determine the probability that in a given minute the server has at lest 3 requests.

IV.6. The number of life insurance sold by a commercial traveller follows a Poisson distribution with a rate of 3 per day. Determine the probability that in a given day he sells at most one life insureance.

IV.7. The number of art exhibitions at the Culture Palace follows a Poisson distribution with a rate of 5 exhibitions per year. Find the probability that, in a given year, there are at most two such exhibitions.

Exercises - Poisson distribution

IV.8*. Let X be a random variable Poisson distributed with parameter $\lambda \in \mathbb{N}^*$. Analyse the monotony of $i \mapsto P\{X = i\}$. What is the maximum of this function?

IV.9*. Let X be a random variable Poisson distributed with parameter λ. Prove that

$$P\{X \text{ is even}\} = \frac{1}{2}\left(e^{\lambda} + e^{-\lambda}\right).$$

(*Hint: Use Taylor series* $e^x = \sum_{k \geqslant 0} \frac{x^k}{k!}$.)

IV.10*. Let X be a random variable Poisson distributed with parameter λ. Compute $\mathbb{E}[X!]$.

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