Probabilities and Statistics - Lecture 2

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Introduction

- In today lecture we will study the influence an already produced event has on other events.
- The notions of *conditioning* and *independence* allow to compute new probabilities from known ones.
- This two concepts are central to probability theory.

Introduction

Example:

- Suppose that we toss two dice and we can clearly observe the value on the first die: 5. Having this piece of information, what is the probability that the sum of the dice is at most 7?
- The reasoning follows: knowing that the first die has value 5, the possible outcomes of the experiment are $(5, 1)$, $(5, 2)$, $(5, 3)$, $(5, 4)$, $(5, 5)$ and $(5, 6)$.
- Any of these elementary events have the same probability: $1/6$; the required probability is $2/6$. ♣

Conditional probability

Definition 1

Let A and B be two random events, the conditional probability *of A given that B has occurred is*

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, (B \neq \varnothing).$$

$P(A|B)$ is also called the probability of A given B.

Example. Two digits are randomly chosen from those between 1 and 9. If their sum is even, what is the probability that one of the chosen digits is even?

Solution: Even digits are $\{2, 4, 6, 8\}$; if the sum is even and one of the digits is even, then both digits are even.

Conditional probability

We have $\binom{4}{2} = 6$ ways of choosing an unordered pair of even digits, and $\binom{5}{2} = 10$ ways of choosing an unordered pair of odd digits (odd digits are $\{1, 3, 5, 7, 9\}$). The probability is

$$\frac{\binom{4}{2}}{\binom{4}{2} + \binom{5}{2}} = \frac{6}{6+10} = 0.375. \clubsuit$$

Example. An urn contains four white balls (two having number 1, two having number 2), five yellow (three having number 1, two having number 2) and six black (two having number 1, four having number 2). A ball is withdrawn from the urn.

Conditional probability

(a) If the ball is not black, what is the probability that the ball is white?

(b) If the ball has number 2, what is the probability that the ball is not white?

Solution:

(a) If the ball is not black, there are nine possible outcomes, and among them four are white balls. The probability is $4/9$.

(b) If the ball has number 2, the sample space becomes: two white balls, two yellow balls and four black balls. The desired probability is $6/8 = 0.75$. ♣

Conditional probability

Example. We toss a fair coin three times in a row. We wish to find the conditional probability $P(A|B)$, where A and B are the events: $A =$" more heads than tails come up", $B =$"first toss gives a head".

Solution: S

$$\Omega = \{hhh, hht, hth, htt, thh, tht, tth, ttt\}.$$

Therefore, $B = \{hhh, hht, hth, htt\}$, $A = \{hhh, hht, hth, thh\}$, $A \cap B = \{hhh, hht, hth\}$.

$$P(B) = \frac{4}{8}, P(A \cap B) = \frac{3}{8}, P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3}{4}.$$

Because all possible outcomes are equally likely here, we can also compute $P(A|B)$ using a shortcut. We can bypass the calculation of $P(B)$ and $P(A \cap B)$, and simply divide the number of elements shared by A and B (which is 3) to the number of elements of B (which is 4), to obtain the same result $3/4$.

Independent events

- In many cases the probability $P(A|B)$ is different from $P(A)$ which is the unconditional probability of event A. We interpret this by saying that event B has a real influence on event A.
- When $P(A) = P(A|B)$ we can say that event A is independent from B (we will see that this property is symmetrical). In other words, A is independent from B if the occurrence of B doesn't change the chances of A to occur.

Definition 2

Two events A and B are called independent *if*

$$P(A \cap B) = P(A) \cdot P(B) \tag{1}$$

Independent events

- If B is a possible event (i.e., $P(B) > 0$) and $P(A|B) = P(A)$, then A and B are independent (but our definition doesn't exclude the possibility that one of the events is impossible).
- The impossible event, $\varnothing$, is independent from any other event. Same property holds for the total event, Ω.
- Independence can be verified by using equation (1), but there are many situation when it is obviously based on the "physical independence" of the two random events.
- The following exercise underscores such a situation.

Independent events - examples

Example. We toss two dice; let A ="the first die has an even value" and B ="the second die has a value at least three". Prove that these two events are independent.

Solution: Intuitively, these two random events are independent because the two values are not related (we can toss the first die long before the second!). Without any calculus we can say that the two events are independent. ♣

- In other situations the independence must be verified because the "physical" independence doesn't show.
- The following examples show such situations.

Independent events - examples

Example. Consider a deck of 52 cards; someone withdraws a card. Let $A=$"the card is a ten" and $B=$"the withdrawn card is a diamond" be two random events. Show that the two events are independent.

Solution: $P(A)=\frac{4}{52}$, $P(B)=\frac{13}{52}=\frac{1}{4}$, $P(A\cap B)=\frac{1}{52}$ and $P(A)\cdot P(B)=\frac{52}{52\cdot 52}=\frac{1}{52}$. ♣

Example. In a box there are four cards: a jack of clubs, a queen of diamonds, a three of spades, and an eight of hearts. A card is withdrawn from the box; let $A=$"the card is a red one" and $B=$"the card is a figure". Analyze the independence of the events A and B.

Solution: We can't assert the independence without computing the probabilities; $P(A)=P(B)=\frac{1}{2}$, $P(A\cap B)=\frac{1}{4}$. ♣

Independent events

Proposition 1

If random events A and B are independent, then the following pairs are also independent $(A, \overline{B})$, $(\overline{A}, B)$ and $\overline{A}, \overline{B})$.

proof: *We know that $P(A \cap B) = P(A) \cdot P(B)$; consider the first pair (the proof is similar for the rest):* $P(A) \cdot P(\overline{B}) = P(A) \cdot [1 - P(B)] = P(A) - P(A \cap B) = P(A \setminus B) = P(A \cap \overline{B})$ ■

Definition 3

Random events $(A_i)_{i \in I}$ are jointly independent *if*

$$P\left(\bigcap_{j=1}^{k} A_{i_j}\right) = P(A_{i_1}) \cdot P(A_{i_2}) \cdot \ldots \cdot P(A_{i_k}),$$

for every subset $\{i_1, i_2, \ldots, i_k\} \subseteq I$, where $k \in \mathbb{N}, k \geqslant 2$.

Jointly independent events

Proposition 2

If random events $(A_i)_{i \in I}$ are jointly independent and (I_1, I_2) is a partition of I, then the random events $(A_i)_{i \in I_1} \cup (\overline{A}_i)_{i \in I_2}$ will be also jointly independent.

proof: (Sketch) Observe first that is enough to prove this result only for finite sets I. We use Proposition 1 and the following property: if $A_1, A_2, \ldots, A_k, A_{k+1}$ are jointly independent, then $A_1, A_2, \ldots, A_k \cap A_{k+1}$ are also jointly independent. The main proof will follow by induction on $|I|$. ■

- The jointly independence is a very strong property and it is difficult to check.
- In many situations jointly independence is verified because of the "physical" independence (for sequential random experiences).

Conditional independence

Proposition 3

The conditional probabilities of events, conditioned on a particular event, say A, form a probability function on a new sample space, A.

proof: (Sketch) Let A be a particular possible random event and define $Q : \mathcal{P}(A) \to [0, 1]$ by $Q(B) = P(B|A)$, for every $B \subseteq A$. Checking that Q satisfies the three axioms of a probability function is left as an exercise. ∎

Definition 4

Given an event, $C \neq \varnothing$, *the events A and B are* conditionally independent *given C if $P(A \cap B|C) = P(A|C) \cdot P(B|C)$.*

Conditional independence

Proposition 4

Given an event, $C \neq \varnothing$, if $B \cap C \neq \varnothing$, then the events A and B are conditionally independent if and only if $P(A|C \cap B) = P(A|C)$.

proof:

$$P(A \cap B|C) = \frac{P(A \cap B \cap C)}{P(C)} = \frac{P(B \cap C) \cdot P(A \cap B \cap C)}{P(B \cap C) \cdot P(C)} =$$

$$= \frac{P(B \cap C)}{P(C)} \cdot \frac{P(A \cap B \cap C)}{P(B \cap C)} = P(B|C) \cdot P(A|C \cap B).$$

Now, A and B are conditionally independent (given C) if and only if $P(B|C) \cdot P(A|C \cap B) = P(A|C) \cdot P(B|C)$, i. e., $P(A|C \cap B) = P(A|C)$.

■

Conditional independence

- The relation $P(A|C \cap B) = P(A|C)$ states that if C is known to have occurred, the additional information that B occurred also doesn't change the probability of A.
- Independence of two random events A and B with respect to the unconditional probability law, does not imply conditional independence, and vice versa.

Total probability formula

Proposition 5

Let $\{A_1, A_2, \dots, A_n\}$ *be a partition of the sample space* $(\bigcup_{i=1}^{n} A_i = \Omega$ *and* $A_i \cap A_j = \varnothing, \forall\, i \neq j)$*. If* B *is a random event, then*

$$P(B) = \sum_{i=1}^{n} P(A_i) \cdot P(B|A_i),$$

where all conditioning events are possible.

proof: $P(B) = P(B \cap \Omega) = P\left[B \cap \left(\bigcup_{i=1}^{n} A_i\right)\right] = P\left[\bigcup_{i=1}^{n} (B \cap A_i)\right]$

$$= \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(A_i) \cdot P(B|A_i).$$

Total probability formula - examples

Example. Urn U_1 contains 3 white balls and 5 black balls, and urn U_2 contains 4 white balls and 6 black balls. A ball is withdrawn from one of the urns which is randomly chosen (the two urns are identical). What is the probability that the withdrawn ball is white?

Solution: Let A_i ="the chosen urn is U_i" ($i = \overline{1,2}$) and B ="the withdrawn ball is white". $A_1 \cup A_2 = \Omega$, $A_1 \cap A_2 = \varnothing$ and $P(A_1) = P(A_2) = 1/2$. We have

$$P(B) = P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2), \text{ and}$$

$$P(B|A_1) = \frac{3}{8}, \; P(B|A_2) = \frac{4}{10}, \text{ therefore}$$

$$P(B) = \frac{1}{2} \cdot \frac{3}{8} + \frac{1}{2} \cdot \frac{4}{10} = \frac{31}{80}. \clubsuit$$

Total probability formula - examples

Example. You roll a fair six-sided die. If the result is 1 or 2, you roll it once more but otherwise, you stop. What is the probability that the total sum of your roll(s) is at least 4?

Solution: Let A_i ="the result of first roll is i" ($i = \overline{1,6}$) and B ="the total sum is at least 4". $\bigcup_{i=1}^{6} A_i = \Omega$, $A_i \cap A_j = \varnothing$, $\forall i \neq j$ and $P(A_i) = 1/6$. We have $P(B) = \sum_{i=1}^{6} P(A_i) \cdot P(B|A_i)$. Given the event A_1, the total sum will be at least 4 if the second roll result is at least 3; given the event A_2 the sum total will be at least 4 if the second roll result is at least 2, hence

$$P(B|A_1) = \frac{4}{6},\ P(B|A_2) = \frac{5}{6}, P(B|A_3) = 0, P(B|A_i) = 1, i = \overline{4,6}$$

$$P(B) = \frac{1}{6} \cdot \left(\frac{4}{6} + \frac{5}{6} + 1 + 1 + 1 \right) = \frac{27}{36}. \clubsuit$$

Bayes formula

Proposition 6

Let $A_1, A_2, \ldots, A_n$ *be a partition of the sample space and* B *a random event, then*

$$P(A_k|B) = \frac{P(A_k)\cdot P(B|A_k)}{\sum\limits_{i=1}^{n} P(A_i)\cdot P(B|A_i)},$$

if all conditioning events are possible. ($P(B|A_k)$ *are* prior probabilities, *and* $P(A_k|B)$ *are* posterior probabilities.)

proof: $$P(A_k|B) = \frac{P(A_k\cap B)}{P(B)} = $$

$$= \frac{P(A_k)\cdot P(B|A_k)}{P(B)} = \frac{P(A_k)\cdot P(B|A_k)}{\sum\limits_{i=1}^{n} P(A_i)\cdot P(B|A_i)}.$$

Bayes formula - examples

Example. A box contains four dice: one has number 3 on exactly four faces, two of them have number 3 on exactly three faces, and one is a regular die. We withdraw at random a die from the box and we toss it once.

(a) Find the probability that we get a 3.

(b) If we get a 3, what is the probability that the withdrawn die was the regular one?

Solution: Define $A_1 =$"the withdrawn die has number 3 on exactly four faces", $A_2 =$"the withdrawn die has number 3 on exactly three faces", $A_3 =$"the withdrawn die is regular", and $B =$"after tossing the die we get a 3".

Bayes formula - examples

$A_1 \cup A_2 \cup A_3 = \Omega$, $A_i \cap A_j = \varnothing$, for any $i \neq j$, and $P(A_1) = P(A_3) = 1/4$, $P(A_2) = 2/4$.
Prior probabilities are

$$P(B|A_1) = \frac{4}{6},\ P(B|A_2) = \frac{3}{6}, P(B|A_3) = \frac{1}{6}.$$

(a) The required probability is:

$$P(B) = \sum_{i=1}^{3} P(A_i)P(B|A_i) = \frac{1}{4} \cdot \frac{4}{6} + \frac{2}{4} \cdot \frac{3}{6} + \frac{1}{4} \cdot \frac{1}{6} = \frac{11}{24}.$$

(b) The required posterior probability is

$$P(A_3|B) = \frac{P(A_3) \cdot P(B|A_3)}{P(B)} = \frac{\frac{1}{4} \cdot \frac{1}{6}}{\frac{11}{24}} = \frac{1}{11}. \clubsuit$$

Bayes formula - examples

Example. We have two urns; first urn contains 3 red, 2 blue, and 3 black balls, the second contains 2 white, 2 blue, and 3 black balls. From the first urn we withdraw a ball and we put it in the second urn. Then, we withdraw a ball from the second urn.

(a) If the second ball is black, what is the probability that the first was blue?

(b) If the second ball is red, what is the probability that the first was blue?

(c) If the second ball is white, what is the probability that the first was red?

Solution: Denote $A_1 =$"the first withdrawn ball is red", $A_2 =$"the first withdrawn ball is blue", $A_3 =$"the first withdrawn ball is black", and $B =$"the second withdrawn ball is black", $C =$"the second withdrawn ball is red", $D =$"the second withdrawn ball is white".

Bayes formula - examples

$A_1 \cup A_2 \cup A_3 = \Omega$, $A_i \cap A_j = \varnothing$, pentru orice $i \neq j$ şi $P(A_1) = 3/8$, $P(A_2) = 2/8$, $P(A_3) = 3/8$.

(a) Prior probabilities are

$$P(B|A_1) = \frac{3}{8},\ P(B|A_2) = \frac{3}{8}, P(B|A_3) = \frac{4}{8}.$$

We use first the total probability formula:

$$P(B) = \sum_{i=1}^{3} P(A_i)P(B|A_i) = \frac{3}{8} \cdot \frac{3}{8} + \frac{2}{8} \cdot \frac{3}{8} + \frac{3}{8} \cdot \frac{4}{8} = \frac{27}{64}.$$

Second we use Bayes formula:

$$P(A_2|B) = \frac{P(A_2) \cdot P(B|A_2)}{P(B)} = \frac{\frac{2}{8} \cdot \frac{3}{8}}{\frac{27}{64}} = \frac{6}{27} = \frac{2}{9}.$$

Bayes formula - examples

(b) Prior probabilities are

$$P(C|A_1) = \frac{1}{8},\ P(C|A_2) = 0, P(C|A_3) = 0.$$

We use first the total probability formula:

$$P(C) = \sum_{i=1}^{3} P(A_i)P(C|A_i) = \frac{3}{8} \cdot \frac{1}{8} + \frac{2}{8} \cdot 0 + \frac{3}{8} \cdot 0 = \frac{3}{64}.$$

Second we use Bayes formula:

$$P(A_2|C) = \frac{P(A_2) \cdot P(C|A_2)}{P(C)} = \frac{\frac{2}{8} \cdot 0}{\frac{3}{64}} = 0.$$

Bayes formula - examples

(c) Prior probabilities are

$$P(D|A_1) = \frac{2}{8},\ P(D|A_2) = \frac{2}{8}, P(D|A_3) = \frac{2}{8}.$$

We use first the total probability formula:

$$P(D) = \sum_{i=1}^{3} P(A_i)P(D|A_i) = \frac{3}{8}\cdot\frac{2}{8} + \frac{2}{8}\cdot\frac{2}{8} + \frac{3}{8}\cdot\frac{2}{8} = \frac{16}{64}.$$

Second we use Bayes formula:

$$P(A_1|D) = \frac{P(A_1)\cdot P(D|A_1)}{P(D)} = \frac{\frac{3}{8}\cdot\frac{2}{8}}{\frac{16}{64}} = \frac{6}{16} = \frac{3}{8}.\clubsuit$$

Seminar's Exercises

- Conditional probability and independence: I.2, I.3, I.6, I.7, I.8, I.15, I.20
- Probabilistic formulas: II.2, II.5, II.7, II.11
- Reserve: I.4, I.9, I.20, II.1, II.8

End

Exercises - Conditional Probability and Independence

I.1. We toss a die and consider the events A: "the value is $1, 2$ or 3" and B: "the value is $2, 3, 4$ or 6". Are A and B are independent?

I.2. We toss two dice and denote by a_1 the value on the first die and by a_2 the value on the second die. Show that events "$a_1 \geqslant 4$" and "$a_2 \leqslant 3$" are independent.

I.3. A high-school graduate sends admission requests to Oxford and Cambridge. He knows that there are 40% chances that Oxford will accept him and 30% chances that Cambridge will accept him. He also knows that there are 20% chances that he will be accepted by both colleges.

(a) If he first receive a consenting letter from Cambridge, what is the probability that Oxford will accept him too?

(b) The events "accepted by Oxford" and "accepted by Cambridge" are compatible? But independent?

Exercises - Conditional Probability and Independence

I.4. We toss three identical coins.

(a) Events "head on the first coin" and "tail on the last two coins" are independent?

(b) Events "head on exactly two coins" and "tail on all coins" are independent?

I.5. Three athletes do practice shooting; the first hits the target with probability $\frac{2}{3}$, the second with $\frac{3}{4}$, and the third with $\frac{4}{5}$ (these random events are jointly independent). What is the probability that the target is hit

(a) by all three athletes?

(b) by exactly two of them?

(c) at least once?

Exercises - Conditional Probability and Independence

I.6. The probability that a student pass an exam is 2/5, for the student on its left this probability is 3/5, and for the student on its right the probability is 1/5. Suppose that the three students doesn't influence each other during the exam. What is the probability that exactly two students pass the exam? But the probability that the middle student pass knowing that the left one passed?

I.7. A box contains three white balls (two numbered with 1 and one numbered with 2) and five black balls (two numbered with 1 and three numbered with 2). Some one randomly withdraws a ball from the box.

(a) If the ball is white, what is the probability that its number is 1?

(b) If the ball is numbered with 2 what is the probability that it is white?

Exercises - Conditional Probability and Independence

I.8. A box contains sixteen balls numbered from 1 to 16, and colored like follows: $\underbrace{1,2,4,5,16}_{white},\underbrace{3,6,7,\ldots 13}_{black},\underbrace{14,15}_{green}$. We withdraw a ball from the box and consider the events. $A=$"the withdrawn ball is black" and $B=$"the ball has a number greater or equal to 10".

Compute the probabilities for the following events: $A|B$, $A|\overline{B}$, $\overline{A}|B$, and $\overline{A}|\overline{B}$.

I.9. Prove that, if A and B are independent events, then

$$\frac{1}{P(A)}+\frac{1}{P(B)}=\frac{P(A)}{P(A\cap B)}+\frac{P(B)}{P(A\cap B)}.$$

(Appropriate denominators are nonzero.)

I.10. Let A and B be possible random events. Show that

(a) $P(\overline{A}\cup\overline{B})=1-P(B)P(A|B)$;

(b) $P(A\cup B|A\cap\overline{B})=P(A|A\cap\overline{B})P(B|\overline{A}\cap B)$;

Exercises - Conditional Probability and Independence

(c) $\dfrac{P(A|A\cup B)}{P(B|A\cup B)} = \dfrac{P(A)}{P(B)}$;

(d) $\dfrac{P(\overline{B}|A)}{P(B)} + \dfrac{P(\overline{A})}{P(A)} = \dfrac{P(\overline{A}|B)}{P(A)} + \dfrac{P(\overline{B})}{P(B)}$.

(e) $\dfrac{P(A\cap C|B)}{P(\overline{A}\cup \overline{C}|B)} = \dfrac{P(A\cap C)}{P(\overline{A}\cup \overline{C})}\dfrac{P(B|A\cap C)}{P(B|\overline{A}\cup \overline{C})}$.

I.11. Consider three random events A_i, $i = \overline{1,3}$, such that

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2) \cdot P(A_3),$$

$$P(\overline{A}_1 \cap A_2 \cap A_3) = P(\overline{A}_1) \cdot P(A_2) \cdot P(A_3),$$

$$P(A_1 \cap \overline{A}_2 \cap A3) = P(A_1) \cdot P(\overline{A}_2) \cdot P(A_3),$$

$$P(A_1 \cap A_2 \cap \overline{A}_3) = P(A_1) \cdot P(A_2) \cdot P(\overline{A}_3).$$

Prove that the three events are jointly independent.

Exercises- Conditional Probability and Independence

I.12. If A, B, and C are jointly independent random events, then A, B, and $\overline{C}$ are jointly independent too.

I.13. Let A_1, A_2, and A_3 ($P(A_3) > 0$) three jointly independent random event. Prove that

(a) $P(A_1 \cap A_2 | A_3) = P(A_1|A_3)P(A_2|A_3)$ and

(b) $P(A_1 \cup A_2 | A_3) = P(A_1|A_3) + P(A_2|A_3) - P(A_1 \cap A_2|A_3)$.

I.14. Assume that the events A_1, A_2, A_3, A_4 are jointly independent and that $P(A_3 \cap A_4) > 0$. Prove that $P(A_1 \cup A_2 | A_3 \cap A_4) = P(A_1 \cup A_2)$.

I.15. Consider two independent fair coin tosses, and the following events: $A =$"first toss is a head", $B =$"second toss is a head" and $C =$"the two tosses have different results". Prove that the three events are pairwise independent but not jointly independent.

Exercises - Conditional Probability and Independence

I.16. We toss two biased coins. The probability that we get two tails is $1/8$, and the probability that we get two tails knowing that we get at least a tail is $3/14$. Find the probabilities of getting tail for each of these two coins.

I.17. Consider two independent rolls of a fair six-sided die, and the following events: $A =$"first roll is 1, 3 or 5", $B =$"first roll is 3, 4 or 5", and $C =$"the sum of the two rolls is 3 or 7". Prove that $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$, but these events are not jointly independent.

I.18*. Give an example of three random events A, B, and C such that $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$ and $P(\overline{A} \cap \overline{B} \cap \overline{C}) \neq P(\overline{A}) \cdot P(\overline{B}) \cdot P(\overline{C})$.

Exercises - Conditional Probability and Independence

I.19. We consider a random experience whose elementary outcomes are equiprobable such that $|\Omega|$ is a prime number. Prove that no two non-trivial events (i. e., $\neq \varnothing, \Omega$) can be independent.

I.20. We roll a die and define $A = \{1, 2\}$, $B = \{2, 4, 6\}$, and $C = \{1, 4\}$.

(a) Prove that A and B are not conditionally independent given C.

(b) Prove that A and B are (unconditionally) independent.

I.21. Let $C_1, C_2, \ldots C_n$ be a partition of the sample space Ω, and A and B two random events. Suppose that A and B are conditionally independent given C_i, for each $1 \leqslant i \leqslant n$ and B is (unconditionally) independent of all C_i's. Prove that A and B are independent. (*Hint: use Proposition 4.*)

Exercises - Total Probability and Bayes Formulas

II.1. We have four identical urns containing: U_1 - 4 white balls and 5 black balls; U_2 - 3 white balls and 7 black balls; U_3 - 2 white balls and 4 black balls; U_4 - 3 white balls and 5 black balls. From a randomly chosen urn we withdraw a ball.

(a) What is the probability that the ball is white?

(b) If the ball is black, what is the probability that it belongs to U_2?

II.2. The probability that a door is locked is $1/2$. The key for this door is on a panel containing twelve keys. We have the right to choose two keys from the panel.

(a) What is the probability that we can open the door?

(b) If we can open the door (using or not one of the two keys) what is the probability that it was locked?

Exercises - Total Probability and Bayes Formulas

II.3. A box contains three coins: the first coin (M_1) has the probability of getting a head $1/4$, the second, (M_2), has this probability equal with $3/4$, and the third coin, (M_3), is a regular one. We withdraw a coin and we toss it once.

a) What is the probability that we get a head?

b) If we get a head what is the probability that the withdrawn coin is M_1?

II.4. We have four six-sided dice: first of them, (D_1), has number 6 on all faces, the second die, (D_2), has number six 6 on three of its faces and number 3 on the other faces, the other dice, (D_3, D_4), are regular. We choose at random a die and we roll it two times.

(a) What is the probability that we get number 6 two times?

(b) If we get number 6 two times what is the probability that the chosen die was D_2?

Exercises - Total Probability and Bayes Formulas

II.5. In a box we have four decks of (52) cards: one of them, (P_1), contains 52s fives of clubs, the second and the third, (P_2, P_3), are regular, and the last, (P_4) contains 52s aces of clubs. We withdraw a deck from the box, and, then we withdraw a card from the chosen deck.

a) What is the probability that we get a number (between 2 and 10) of clubs?

b) If we get an ace, what is the probability that the chosen deck was P_2 or P_3?

II.6. In a box we have three coins: one has two heads, one has two tails, and the third is a regular one. We randomly withdraw a coin from the box and toss it.

(a) What is the probability that we get a tail?

(b) If we get a tail, what is the probability that the coin was the regular one?

Exercises - Total Probability and Bayes Formulas

II.7. We withdraw a ball from a box containing one white ball and two red balls. If the ball is white, then we put it back in the box together with another white ball; If the ball is red, then we put it back in the box together with two other red balls. After these operations we withdraw another ball from the box.

(a) What is the probability that the second withdrawn ball is red?
(b) If the second ball is red, what is the probability that the first was red?

II.8. A programmer asks a recommendation letter to its former employer. He estimates his chances to get a new job at: 50% having at hand a strong recommendation, 40% with a regular recommendation and 20% with a lousy one. He also knows that its former boss will give him a strong recommendation with probability 0.4, and a regular one with probability 0.4.

(a) What is the probability that the programmer will get a new job?

Exercises - Total Probability and Bayes Formulas

(b) If he gets a new job, what is the probability that its recommendation was lousy?

II.9. A box contains three dice: two of them have number 5 on five of their faces, while the remaining one has number 5 on four of its faces. We withdraw at random a die and we roll it.

(a) What is the probability that we get a 5?

(b) If we got a 5, what is the probability that the withdrawn die was that with number 5 on four of its faces?

II.10. We have two urns: U_1 has 3 white, 3 red, and 4 black balls, U_2 contains 2 white, 3 red, and 2 black balls. From U_1 we withdraw a ball which is placed in U_2, then we withdraw a ball from U_2.

(a) What is the probability that we get a white ball at the second withdrawal?

(b) If at the second withdrawal we got a white ball, what is the the probability that we got a red ball at the first withdrawal?

Exercises - Total Probability and Bayes Formulas

II.11. A box contains two coins: a regular coin and one fake two-headed coin. We choose a coin at random and toss it twice. Define the following events: $A =$"first toss shows head", $B =$"second toss shows head", and $C =$"the chosen coin is the regular one".

(a) Prove that A and B are independent given C.

(b) Compute $P(A|C), P(B|C)$, and $P(A \cap B|C)$.

(c) Compute $P(A), P(B)$, and $P(A \cap B)$.

(d) Are A and B independent?

(*Hint: use Proposition 4.*)

Exercises - Total Probability and Bayes Formulas

II.12. Let A and B two possible random events. We say that A *suggests* B, if $P(A|B) > P(A)$ and that A *doesn't suggest* B if $P(A|B) < P(A)$.

(a) Prove that A suggests B if and only if B suggests A.

(b) If $P(\overline{A}) > 0$, then A suggests B if and only if $\overline{A}$ doesn't suggest B.

(c) We know that a treasure is located in one of two places, with probabilities $\beta \in (0,1)$ and $(1-\beta)$, respectively. We search the first place and if the treasure is there, we find it with probability $p > 0$. Show that the event of not finding the treasure in the first place suggests that the treasure is in the second place.

Exercises - Total Probability and Bayes Formulas

II.13*. A particular class has had a history of low attendance. The annoyed professor decides that she will not lecture unless at least k of the n students enrolled in the class are present. Each student will independently show up with probability p_g if the weather is good, and with probability p_b if the weather is bad. The probability of bad weather on a given day is q.

(a) Calculate the probability that the professor will teach her class on a particular day.

(b) If he lectures in a particular day, what is the probability that the weather was bad?

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