Probabilities and Statistics - Lecture 1

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The Structure of the Lectures, Scores and Grades

- The first six weeks are dedicated to the (Discrete) Theory of Probabilities (6 lectures and 6 seminars), while the last seven weeks will concern the (Continuous) Theory of Probabilities and Statistics (7 lectures and 7 laboratories).
- We will have 6 seminars, 7 laboratories and 13 lectures.
- From seminars and labs you will get two intermediate scores T_1 and T_2 as follows:

The Structure of the Lectures, Scores and Grades

- T_1 comes from six tests one on each seminar (15 minutes). Those who fail to receive at least 30 points (from a maximum of $6 \times 10 = 60$ points) cannot pass the course ($T_1 \geqslant 30$).
- T_2 comes from the exercises solved in class (20 points), from the homeworks (20 points, deadline in the 12th week), and a test given in the final week (20 points). This score also must be at least 30 points (from a maximum of $20 + 20 + 20 = 60$ points). Those who fail to receive 30 points cannot pass the course ($T_2 \geqslant 30$).

There will be no arrears or enhancements session!

The Structure of the Lectures, Scores and Grades

- The final score

$$T = T_1 + T_2.$$

- This scores will be transformed in grades like follows: $T \in [60, 70) \to$ grade 5, $T \in [70, 80) \to$ grade 6, $T \in [80, 90) \to$ grade 7, $T \in [90, 100) \to$ grade 8, $T \in [100, 110) \to$ grade 9 şi $T \in [110, 120) \to$ grade 10.
- Both intermediate scores (T_1 and T_2) must be at least 30 points.
- If $T_1 < 30$ and/or $T_2 < 30$, then the course will not be considered passed.

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Random experience

- Intuitively, a *random experience* corresponds to a process producing a result which is not known before performing this process; what is known is the set of all possible results.
- The *outcomes* of such an experience are registred by a neutral observer. Usually, random experiences are practical ones, but can be also abstract.
- The outcome of a random experience are due to *chance* only. At a given moment we will consider only one random experience (even if this experience can be re-performed).

Random experience

Definition 1

A random experience is an experience whose result is not known before peforming it, but whose all possible results are known, and which can be performed in the same conditions at any given time.

Examples:

- By tossing a die the possible outcomes are $\{1, 2, \ldots, 6\}$, but we don't know for sure which number will occur. ♣
- By tossing two dice the possible outcomes are:

$$\{(1,1),(1,2),\ldots,(6,6)\}. ♣$$

- By flipping two coins, the outcomes are $\{(H,H), (H,T), (T,H), (T,T)\}$. ♣
- If the experience is to measure the lifespan of a battery the outcome will be a real non-negative number. ♣

Elementary random events

- Following a random experience means to observe its outcome; each time we will have only one outcome (from the known set of all possible results). We say in this case that an *elementary event* is produced.

Definition 2

A possible result of a random experience is called an elementary random event*, the set of all these events is the* sample space*, denoted by* Ω.

- After performing an experience only one elementary event from Ω "happens" - following the "chances" each outcome has.
- In the process of making reasoning about the experience, our interest may be directed not for the outcome itself but for a family to which it belongs.

Random events

- The above observation extends the notion of elementary random event.

Definition 3

A random event *is a certain subset of the sample space:* $A \subseteq \Omega$.

- We toss a die: the sample space is $\Omega = \{1, 2, \ldots, 6\}$; if we are interested in even numbers, the corresponding random event is $A = \{2, 4, 6\} \subseteq \Omega$. ♣
- We toss two dice: $\Omega = \{(1, 1), (1, 2), \ldots, (6, 6)\}$; if we are interested in a sum equal with 4, then the corresponding random event is $A = \{(1, 3), (2, 2), (3, 1)\} \subseteq \Omega$. ♣
- If we flip two coins: $\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$. The random event: "the two sides are different" is $A = \{(T, H), (H, T)\}$. The event "at least one of the two sides is head" is $B = \{(H, H), (H, T), (T, H)\}$. ♣

Random events

- Usually a random event is defined using a predicate: the event will be formed with all the elementary random events which make the predicate true.
- Formally, a random event is a subset of the sample space Ω.
- If Ω is a discrete set (i. e., $|\Omega| \leqslant \aleph_0$), then any subset of $A \subseteq \Omega$ can be considered a random event.
- If Ω is not discrete, then only specific subsets of Ω can be random events - this is not the subject of our lectures (about discrete probabilities).

Random events - properties and notations

Definition 4

We say that a random event $A \subseteq \Omega$ occurs *if, after performing the random experience, the outcome (the elementary random event) belongs to* A.

- We denote random events using big caps like A, B, C etc. Because in our applications Ω will be a discrete set, we can consider that any subset of Ω is a random event.
- From now on we do not make any distinction between subsets of Ω and random events; operations and properties over subsets are transfered over random events.

- $\varnothing$ is the impossible event (it will never occur);
- Ω is the total event (it occurs everytime);

Random events - properties and notations

- if $A, B \subseteq \Omega$ are random events, then
  - $A \cup B$ is a random event which occurs if and only if one of the events A or B occurs;
  - $A \cap B$ is a random event which occurs if and only if both events A and B occurs;
  - $A \Delta B$ is a random event which occurs if and only if exactly one of the events A and B occurs;
  - $A \setminus B$ is a random event which occurs if and only if A occurs but B doesn't;
- if A is a random event, then $\overline{A} = \Omega \setminus A$ is a random event called the complementary of A (or contrary to A): $\overline{A}$ occurs if and only if A doesn't occur;
- if $A \subseteq B$, we say that A implies B;

Random events - properties and notations

- if $A \cap B = \varnothing$, then we say that A is incompatible with B (or disjoint of B); if $A \cap B \neq \varnothing$, then A is compatible with B;
- if $A_i \cap A_j = \varnothing$, for any two distinct random events from a family $(A_i)_{i \in I}$, then we say that this family is composed of mutually incompatible (or pairwise incompatible/disjoint random events; we also say that the events are mutually exclusive);
- in a similar manner we can define the union or the intersection of a finite (or infinitely countable) family of events.

Example. Consider the tossing of two dice. Let A be the event "the sum of the dice is 5" and B = "at least one die is greater or equal with 4":

$$A = \{(1,4),(2,3),(3,2),(4,1)\};$$

$$B = \{(1,4),(2,4),\ldots,(6,4),(1,5),(2,5),\ldots,(6,5),(1,6),(2,6),\ldots,(6,6),$$
$$(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),(6,1),(6,2),(6,3)\}.$$

Probability function

Then

- $A \cup B = \Omega \setminus \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1),(3,3)\}$;
- $A \cap B = \{(1,4),(4,1)\}$ - A and B are compatible;
- $A \Delta B = (A \cup B) \setminus (A \cap B) = \dots$
- $\overline{A} = \{(1,1),(1,2),(2,1),(3,2),(2,3),\dots(6,6)\}$. ♣

- We will consider Ω to be at most infinitely countable:

$$\Omega = \{\omega_1, \omega_2, \dots\}.$$

- The concept of *probability* comes from the idea of associating a real number to each random event in order to compare their chances.
- The definition is based on the empirical observation that an event which occurs more often must have a greater probability than one which occurs rarely.
- The probability of a random event A is denoted by $P(A)$.

Axioms of probability function

- The following axioms define a *probability function*. Note that on the same set Ω we can define different probability functions!

Definition 5

(Kolmogorov) If Ω is the sample space of a random experience, a probability function *on Ω is a function P defined on the random events family (in our case $\mathcal{P}(\Omega) = 2^{\Omega}$) which satisfies*

- Axiom 1. $0 \leqslant P(A) \leqslant 1$, *for any random event A.*
- Axiom 2. $P(\Omega) = 1$ *and* $P(\varnothing) = 0$.
- Axiom 3. *If events $A_1, A_2, \ldots, A_k, \ldots$ are mutually exclusive, then*

$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k).$$

Axioms of probability function

- *Axiom 1* says that the probability asociated with a random event is a real number from $[0,1]$.
- *Axiom 2* says that any outcome of the experience belongs, with probability 1, to Ω. Moreover, it says that the impossible event doesn't occur.
- *Axiom 3* says that for a mutually incompatible sequence of random events, the probability of its union is the sum of the probabilities of all these events.
- If the sequence is infinite, *Axiom 3* guarantees the convergence of the corresponding series from the right member.

Properties of the probability function

Proposition 1

Let $A_1, A_2, \ldots, A_m$ be a finite sequence of mutually incompatible random events, then $P\left(\bigcup_{k=1}^{m} A_k\right) = \sum_{k=1}^{m} P(A_k)$.

proof: Consider in Axiom 3, $A_k = \varnothing, \forall\, k \geqslant m+1$. ■

Proposition 2

Let A and B be two random events, then

(i) $P(A) = P(A \cap B) + P(A \setminus B)$;

(ii) $P(A \cup B) = P(A) - P(A \cap B) + P(B)$;

(iii) $P(\overline{A}) = 1 - P(A)$;

(iv) *if A implies B $(A \subseteq B)$, then $P(A) \leqslant P(B)$* .

Properties of the probability function

proof: (i): in Proposition 1 we can take $m = 2$, $A_1 = A \cap B$, $A_2 = A \setminus B$.. (ii): we use (i) and Proposition 1. (iii): we can use Proposition 1 and Axiom 2: $1 = P(\Omega) = P(A) + P(\overline{A})$. (iv): use (i) and Axiom 1 and get $P(B) = P(B \cap A) + P(B \setminus A) = P(A) + P(B \setminus A) \geqslant P(A)$. ■

Proposition 3

(Inclusion-exclusion principle) If $A_1, A_2, \ldots, A_m$ are random events, then

$$P\left(\bigcup_{k=1}^{m} A_k\right) = \sum_{k=1}^{m} P(A_k) - \sum_{1 \leqslant k_1 < k_2 \leqslant m} P(A_{k_1} \cap A_{k_2}) + \cdots +$$

$$+(-1)^{p+1} \sum_{1 \leqslant k_1 < k_2 < \ldots < k_p \leqslant m} P\left(A_{k_1} \cap A_{k_2} \cap \ldots \cap A_{k_p}\right) + \cdots +$$

$$+(-1)^{m+1} P(A_1 \cap A_2 \cap \ldots \cap A_m).$$

Properties of the probability function

Demonstrateţie: We use induction on m: for $m = 2$ we have property 2, (ii). Suppose now that the above relation holds for any $(m-1)$ random events.

$$P\left(\bigcup_{k=1}^{m} A_k\right) = P\left(\bigcup_{k=1}^{m-1} A_k\right) - P\left[\left(\bigcup_{k=1}^{m-1} A_k\right) \cap A_m\right] + P(A_m) = \quad (1)$$

$$= P\left(\bigcup_{k=1}^{m-1} A_k\right) - P\left[\bigcup_{k=1}^{m-1} (A_k \cap A_m)\right] + P(A_m).$$

We use the inductive hypothesis for the first two terms:

$$P\left(\bigcup_{k=1}^{m-1} A_k\right) = \sum_{k=1}^{m-1} P(A_k) - \sum_{1 \leqslant k_1 < k_2 \leqslant m-1} P(A_{k_1} \cap A_{k_2}) + \cdots + \quad (2)$$

$$+(-1)^{p+1} \sum_{1 \leqslant k_1 < k_1 < \ldots < k_p \leqslant m-1} P\left(A_{k_1} \cap A_{k_2} \cap \ldots \cap A_{k_p}\right) + \cdots +$$

Properties of the probability function

$$+(-1)^m P(A_1 \cap A_2 \cap \ldots \cap A_{m-1}),$$

$$P\left[\bigcup_{k=1}^{m-1}(A_k \cap A_m)\right] = \sum_{k=1}^{m-1} P(A_k \cap A_m) - \tag{3}$$

$$= \sum_{1\leqslant k_1<k_2\leqslant m-1} P(A_{k_1} \cap A_{k_2} \cap A_m) + \cdots +$$

$$+(-1)^{p+1} \sum_{1\leqslant k_1<k_1<\ldots<k_p\leqslant m-1} P\left(A_{k_1} \cap A_{k_2} \cap \ldots \cap A_{k_p} \cap A_m\right) + \cdots +$$

$$+(-1)^{m+1} P(A_1 \cap A_2 \cap \ldots \cap A_{m-1} \cap A_m).$$

Combining (1), (2) and (3) we get the desired relation. ■

Probability of elementary random events

- Let $P(\omega)$ be the probability of the elementary random event $\{\omega\}$, $\forall\, \omega \in \Omega$. From Axiom 3, for every random event A, we have

$$P(A) = \sum_{\omega \in A} P(\omega). \tag{4}$$

- If A contains a finite number of elements, then the above sum is finite; if A contains an infinitely countable number of elements, then the sum is an infinite series.
- The sum is zero if A is the impossible event.
- We get an equivalent definition if we replace Axiom 3 with

  Axiom 3'. For every random event A,

$$P(A) = \sum_{\omega \in A} P(\omega).$$

Equiprobable elementary random events

- In many situations Ω consists of elementary random events having the same probability:

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}, P(\omega_i) = p, \forall 1 \leqslant i \leqslant n,$$

then

$$P(\Omega) = 1 = \sum_{i=1}^{n} P(\omega_i) = np, \text{ i. e., } P(\omega_i) = p = \frac{1}{n}, \forall i.$$

- Let us consider, in this case, a random event A, with $|A| = k$, $A = \{\omega_{i_1}, \omega_{i_2}, \ldots, \omega_{i_k}\}$, then

$$P(A) = \sum_{j=1}^{k} P(\omega_{i_j}) = \frac{k}{n} = \frac{|A|}{|\Omega|}.$$

Equiprobable elementary random events

We just proved

Proposition 4

If Ω is finite and the elementary random events are equiprobable, then the probability of a random event A is the ratio between the cardinality of A and the cardinality of Ω.

- In this situation the probability of a random event is the number of "favorable" cases over the total number of cases. Computing probabilities becomes a counting problem.
- Situations in which we can apply the above theoretical result must be carefuly distinguished as *the elementary random events are not always equiprobable*. As a witness we can use the simmetry priciple: similar events must have equal probabilities.

Equiprobable elementary random events

Example. When we withdraw a ball from a box, any ball has the same probility to be chosen. If the box contains balls of different colors: two white and three black, a black ball has a different probability of occurrence than a white ball. ♣

Example. If we toss a fair die, the probability of occurrence of a number is always the same for all the numbers from $\{1, 2, \ldots, 6\}$. But if three faces of the die have number 1, and the remaining have 2, 3, and 4 respectively, then the elementary outcomes have different probabilities. ♣

Seminar's Exercises

- Random Events: I.1, I.3, I.5
- Probability function: II.1 (b,c), II.2 (c), II.3 (or II.4), II.5 (or II.6), II.8 (or II.9), II.14
- Reserve: II.12, II.18, II.19

End

Exercises - Random Events

I.1. Three players, 1, 2, and 3 flip (in this order) a coin. The winner is the player which gets the tail first. The set of elementary random events is (0 for head and 1 for tail)

$$\Omega = \{1, 01, 001, 0001, \ldots, 0000\cdots\}$$

a) Determine the following random events: A_i = "player i wins the game, $i = \overline{1,3}$".

b) Determine the following random events $\overline{A_1 \cup A_3}$, $\overline{A_1} \cup A_2$, $A_1 \cup A_2 \cup A_3$, and $\overline{A_1 \cup A_2 \cup A_3}$.

c) Show that $A_1 \cup A_3 \subseteq \overline{A_2}$. True or false: $A_1 \cup A_3 = \overline{A_2}$?

Exercises - Random Events

I.2. Two dice are tossed (one red and one black). Let A = "the sum is odd", B = "at least one die is 1, and C ="the sum is 5". Describe the following random events using elementary outcomes $A \cap B$, $A \cup C$, $B \cap C$, $A \cap \overline{C}$, $\overline{A} \cap B$, and $A \cap B \cap \overline{C}$.

I.3. Let A and B be two random events associated with a random experience. Show that

a) $\overline{A \cup B} = \overline{A} \cap \overline{B}$, $\overline{A \cap B} = \overline{A} \cup \overline{B}$;

b) $A \cup \overline{B} = \overline{\overline{A} \cap B}$, $A \cap B = B \cap (A \cup \overline{B})$;

c) $A \cap B = B \cap \overline{(\overline{A} \cap B)}$, $A \cap B = B \setminus (\overline{A} \cap B)$.

Exercises - Random Events

I.4. Let A and B be two random events associated with a random experience. Reduce as much as possible the following expressions

a) $(A \cup B) \cap (A \cup \overline{B})$;

b) $(A \cup B) \cap (A \cup \overline{B}) \cap (\overline{A} \cup B)$.

I.5. Let A, B, and C be three random events associated with a random experience. Determine (as expression in A, B, and C) the following random events

a) only A occurs among the three events;

b) A and C occur but not B;

c) at least two of the three events occur;

d) exactly one of the three events occurs;

e) at most three of them occur.

Exercises - Random Events

I.6. Let A and B random events.

a) Show that $\overline{A} = (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B})$ and $\overline{B} = (A \cap \overline{B}) \cup (\overline{A} \cap \overline{B})$;

b) Show that $\overline{A \cap B} = (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B}) \cup (A \cap \overline{B})$.

c) Consider rolling a six-sided die. Let A be the set of outcomes where the roll is an odd number. Let B be the set of outcomes where the roll is less than 4. Compute the sets on both sides of the equality in part b), and verify that the equality holds.

Exercises - Probability Function

II.1. Let A and B be two random events. Show that

a) $P(\overline{A} \cap \overline{B}) = 1 - P(A) - P(B) + P(A \cap B)$;

b) $P(A \cup B) = P(A \setminus B) + P(A \cap B) + P(B \setminus A)$;

c) $P(A \Delta B) = P(A) - 2P(A \cap B) + P(B)$.

II.2. Prove the following identities (A, B, and C are three random events)

a) $P(A \cap B) + P[(A \setminus B) \cup (B \setminus A)] + P(\overline{A} \cap \overline{B}) = 1$;

b) $P(\overline{A} \cap \overline{B}) + P(A) + P(\overline{A} \cap B) = 1$;

c) $P(\overline{A} \cap \overline{B} \cap \overline{C}) + P(A) + P(\overline{A} \cap B) + P(\overline{A} \cap \overline{B} \cap C) = 1$;

d) $P(A \cap B) - P(A)P(B) = P(\overline{A} \cap \overline{B}) - P(\overline{A})P(\overline{B})$.

Exercises - Probability Function

II.3. We consider a random experience with $\Omega = \{\omega_1, \omega_2, \ldots, \omega_6\}$. We know that $P(\{\omega_1, \omega_2, \omega_3\}) = 1/2$, $P(\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}) = 5/6$ and $P(\{\omega_1, \omega_2\}) = 1/6$. We define the following random events $A = \{\omega_1, \omega_2, \omega_4, \omega_5\}$, $B = \{\omega_3, \omega_4, \omega_5, \omega_6\}$ and $C = \{\omega_3, \omega_4, \omega_5\}$.

a) Events A and B are compatible? What about A and C?

b) Compute $P(A \cup (B \setminus C))$, $P(A \Delta B)$, $P(A \cup C)$ and $P(B \setminus A)$.

c) It is possible that $P(\{\omega_2, \omega_3, \omega_5, \omega_6\}) = 3/8$ and $P(\{\omega_1, \omega_2, \omega_4\}) = 15/16$?

II.4. We consider a random experience having $\Omega = \{\omega_1, \omega_2, \ldots, \omega_6\}$. We know that $P(\{\omega_1\}) = P(\{\omega_2\}) = 1/6$ and $P(\{\omega_2, \omega_3, \omega_4, \omega_5\}) = 2/3$. Let $A = \{\omega_3, \omega_4, \omega_5\}$; define a random event B that has to be incompatible with (but not contrary to) A.

a) Prove that $P(A \cup B) < 1$.

b) Compute $P(A \setminus B)$, $P(A \cap B)$, $P(\overline{B})$ and $P(A \Delta B)$.

c) It is possible that $P(\{\omega_1, \omega_2, \omega_3\}) = 1/4$, $P(\{\omega_2, \omega_4, \omega_5, \omega_6\}) = 2/3$, and $P(\{\omega_1, \omega_3\}) = 1/5$?

Exercises - Probability Function

II.5. Let A and B be two random events associated with a random experience such that $P(B \setminus A) = 1/3$, $P(A \cap B) = 1/6$ and $P(A \setminus B) = 1/3$.

a) Compute $P(A)$, $P(A \Delta B)$, $P(A \cup B)$ and $P(A \setminus \overline{B})$.

b) Events A and B are complementary? A and $\overline{B}$ are compatible?

II.6. Let A and B be two random events associated with a random experience such that $P(A \setminus B) = 1/6$, $P(A \cap B) = 1/6$ and $P(B \setminus A) = 1/3$.

a) Compute $P(A \cup B)$, $P(A \Delta B)$, $P(\overline{A} \cap B)$ and $P(\overline{A} \setminus B)$.

b) Events A and B are compatibile? A and $B \setminus A$ are complementary?

II.7. Let A and B be two random events such that $P(A \setminus B) = 2/7$, $P(A \cap B) = 1/7$ and $P(A \Delta B) = 5/7$.

a) Calculate $P(A)$, $P(B)$, $P(A \cup B)$, $P(\overline{A} \setminus B)$ and $P(B \setminus A)$.

b) Events A and B are complementary? But compatible?

Exercises - Probability Function

II.8. We toss two dice and denote: $A =$"the sum of the values is a prime number" and $B =$"the product of the values is $\geqslant 19$".

a) Compute $P(A)$, $P(B)$, $P(A \Delta B)$, $P(A \cup B)$, and $P(A \setminus B)$.

b) The random events A and B are contrary to each other? Are they compatible?

(c) Define a random event which has to imply A and to be incompatible with B.

II.9. We toss two dice and denote: $A =$"the product of the values is $\leqslant 10$" şi $B =$"at least one value is a prime number".

a) Compute $P(A)$, $P(\overline{B})$, $P(A \Delta B)$, $P(A \cap B)$, and $P(B \setminus A)$.

b) The random events A and B are contrary to each other? Are they compatible?

(c) Define a random event which has to be compatible with A but not with B.

Exercises - Probability Function

II.10. Let A and B be two disjoint random events with $P(A) = 0.3$ and $P(B) = 0.5$. What is the probability of the event

a) exactly one of the two events occur?

b) B occur but A doesn't ?

c) both events occur?

II.11. Two senior, s_1 and s_2 and three juniors j_1, j_2 and j_3 are in a chess tournament. Any senior is twice as likely to win as any junior.

a) Find the chances that a junior wins the tournament.

b) What is the probability that s_1 or j_1 wins the tournament?

Exercises - Probability Function

II.12. Out of the students in a class, 60% are geniuses, 70% love chocolate, and 40% fall into both categories. Determine the probability that a randomly selected student is neither a genius nor a chocolate lover.

II.13. US president signs in to law a resolution only if this resolution is voted in both Congress and Senate. 60% of all resolution pass the Congress, 70% pass the Senate, and 80% pass at least of one of the two chambers. Compute the probability that

a) a certain resolution comes to president,

b) a certain resolution passes exactly one of the two chambers.

Exercises - Probability Function

II.14. A student has to choose two of the following three facultative courses: French, Maths, and History. He chooses History with probability 5/8, French with probability 5/8, and chooses both History and French with probability 1/4. What is the probability that he will choose Maths? But the probability that he will choose History or French?

II.15. Three horses compete in a race: H_1, H_2, and H_3. H_1 has twice the chances to win compared to H_2, and H_2 twice the chances to win compared to H_3.

a) What are the probabilities of winning of the three horses?

b) What is the probability that the first or the second horse doesn't win?

Exercises - Probability Function

II.16*. Two students X and Y follow a course for which one could get one of the three grades: A, B or C. The probability that X gets B is 0.3; the probability that Y gets B is 0.4. The probability that neither get A but at least one gets B 0.1. What is the probability that at least one gets B but none of them gets C?

II.17*. In a box there are ten balls numbered from 1 to 10; we randomly select two balls from the box. What is the probability that the sum of the numbers on the balls is odd if

a) the two balls are simultaneously selected?
b) the two balls are selected one by one without removing?
c) the two balls are selected one by one and removed?

Exercises - Probability Function

II.18*. For a biased die the probability of occurrence of a face is proportional with the number on that face. We toss the die.

a) What is the probability that we get an even number?

b) What is the probability that we get a prime number?

II.19**. Let A_1, A_2 and A_3 be three random events.

a) When the following relation holds

$$P(A_1 \cap (A_2 \cup A_3)) = P(A_1 \cap A_2) + P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3)?$$

b) Show that, if $A_1 \cap A_2 \cap A_3 = \varnothing$, then

$$P[(A_1 \cup A_2) \cap (A_2 \cup A_3) \cap (A_3 \cup A_1)] =$$

$$= P(A_1 \cap A_2) + P(A_2 \cap A_3) + P(A_3 \cap A_1).$$

Exercises - Probability Function

II.20. A six-sided die is loaded in a way that each even face is twice as likely as each odd face. All even faces are equally likely, as are all odd faces. Find the probability that the outcome is

a) an odd number;

b) less than 4;

c) an even prime number.

II.21*. If $P(A) = 0.9$ and $P(B) = 0.8$, show that $P(A \cap B) \geqslant 0.7$.

a) Prove that $P(A \cap B) \geqslant P(A) + P(B) - 1$.

b) Prove by induction the Bonferroni inequality: given n random events $A_1, A_2, \ldots A_n$ we have

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geqslant \sum_{i=1}^{n} P(A_i) - n + 1.$$

Exercises - Probability Function

II.22*. Let us consider a random experience having an infinitely countable elementary results:

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_n, \ldots\}.$$

a) Show that is not possible that all elementary random events have the same positive probability p.

b) True or false: all elementary random events have positive probabilities?

II.23*. Consider a random experience having the sample space Ω. Prove that $d : \mathcal{P}(\Omega) \times \mathcal{P}(\Omega) \to [0,1]$, defined by $d(A,B) = P(A\Delta B)$, is a metric on the family $\mathcal{P}(\Omega)$.

Exercises - Probability Function

II.24*. Let $A_1, A_2, \ldots, A_n$ be random events having the following properties:

(i) $A_i \subseteq \bigcup_{j \neq i} A_j, \forall i = \overline{1, n}$;

(ii) $A_i \cap A_j \cap A_k = \varnothing, \forall 1 \leqslant i < j < k \leqslant n$.

Prove that

$$2P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i).$$

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