# Probabilities and Statistics - Lecture 6

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## Table of contents

## Introduction



This chapter is reserved to a concept broadly used in today science (from statistical physics to economic science): *random* or *stochastic processes* (i.e., which randomly changes). Informally a random process is a mathematical model of a random experiment which evolves in time and produces a sequence of numerical values.

A random process can model:

- price variation on stock market;
- succesive positions on radar of a comercial airplane;
- loading variations of a communication node etc.

# Introduction



**Definition 1**

*A* stochastic process *is a family of random variables* $(X(i))_{i \in I}$, *defined over a space with probability.*

- Each variable $X_i = X(i) : \Omega \to \mathbb{R}$ represents a step of the process; if the set $I$ is a discrete one, then we have a *discrete stochastic process*. In what follows we suppose that $|I| \leqslant |\mathbb{N}^*|$.
- The most known stochastic processes:

1. *Arrival processes:* systems of messages, clients who reach a server etc. This type is emphasized by *Bernoulli processes* and by *Poisson processes* which are the countinuous version of the former.
2. *Markov processes:* are probabilistic experiments which evolves in time and for which future steps depend somehow on former ones.

# Discrete Markov Chains



- A Markov chain is a process whose future steps depends on past steps.
- This influence of the past on the future is modeled using process states; these states change in accordance with some known probabilities.
- We will study only the processes wich have a finite number of states.

## Discrete Markov Chains



### Definition 2

(i) *A* discrete Markov chain *with a finite number of states is a stochastic process* $(X_n)_{n\geqslant 1}$, *where* $X_n : \Omega \to S = \{s_1, s_2, \ldots, s_m\}$ *are random variables having the* *Markov property:*

$$P\{X_{n+1} = s | X_1 = s_{i_1}, X_2 = s_{i_2}, \ldots, X_n = s_{i_n}\} =$$

$$= P\{X_{n+1} = s | X_n = s_{i_n}\}, \forall n \geqslant 1.$$

(ii) *A* Markov chain *is called* homogenuous *(or* stationary*) if*

$$P\{X_{n+1} = s_j | X_n = s_i\} = P\{X_n = s_j | X_{n-1} = s_i\} = p_{ij},$$

$$\forall\, n \geqslant 2, s_i, s_j \in S.$$

## Discrete Markov Chains



- In the following we will consider only homogenuous discrete Markov chains having a finite number of states.
- $S$ is the *states space*, and $p_{ij}$ are the *transition probabilities*, the matrix formed with these probabilities, $P = (p_{ij})_{1 \leqslant i,j \leqslant m}$, is called the *transition probability matrix* of the chain.

Such a Markov chain is specified by:

- the set of states $S = \{s_1, s_2, \ldots, s_m\}$;
- the transition probabilities $p_{ij}$.

- A Markov chain can be represented by the *transition probability digraph*: the possible states are the vertices, and the transitions are the arcs (labeled with probabilities).

# Discrete Markov Chains - Examples



*Example.* Alice is taking a probability class and at the end of each week she can be either up-to-date or she may have fallen behind. If she is up-to-date in a given week, the probability that she will be up-to-date (or behind) in the next week is 0.8 (or 0.2, respectively). If she is behind in a given week, the probability that she will be up-to-date (or behind) in the next week is 0.4 (or 0.6, respectively). Alice is (by default) up-to-date when she starts the class.

We have an homogenuous discrete Markov chain with two possible states: $s_1$ - Alice is up-to-date and $s_2$ - she is behind. The transition probabilities are

$$p_{11} = 0.8, p_{12} = 0.2, p_{21} = 0.6, p_{22} = 0.4. \clubsuit$$

# Discrete Markov Chains - Examples



*Example.* A bee moves along a straight line in unit increments. At each time period, it moves one unit to the left with probability 0.3, one unit to the right with probability 0.3, and stays in place with probability 0.4, independently of the past history of movements. Two spiders are lurking at positions 1 and $m$: if the bee lands there, it is captured by a spider, and the process terminates.

We want to define a Markov chain model, assuming that the bee starts in a position between 1 and $m$. The states are $1, 2, \ldots, m$. The transition probabilities are:

$$p_{11} = p_{mm} = 1,$$

$$p_{ij} = \begin{cases} 0.3, & \text{if } j \in \{i-1, i+1\}, \\ 0.4, & j = i \end{cases}, \text{ pentru } i = \overline{2, m-1}. \clubsuit$$

## Path Probabilities and $n$-steps Transitions

- For a given Markov chain we can determine the probability of a sequence of future states using the *multiplication rule*.

**Proposition 1**

*Let $(X_i)_{i\in\mathbb{N}^*}$ be a discrete homogenuous Markov chain with a finite number of states, then*

$$P\{X_1 = s_{i_1}, X_2 = s_{i_2}, \dots, X_n = s_{i_n}\} = P(X_1 = s_{i_1})\cdot p_{i_1 i_2}\cdot p_{i_2 i_3}\cdot\ldots\cdot p_{i_{n-1} i_n}.$$

proof:

$$P\{X_1 = s_{i_1}, X_2 = s_{i_2}, \dots, X_n = s_{i_n}\} = P(X_1 = s_{i_1})\cdot P(X_2 = s_{i_2} | X_1 = s_{i_1})\cdot$$
$$\ldots\cdot P\{X_n = s_{i_n} | X_1 = s_{i_1}, X_2 = s_{i_2}, \dots, X_{n-1} = s_{i_{n-1}}\} =$$
$$= P(X_1 = s_{i_1})\cdot p_{i_1 i_2}\cdot p_{i_2 i_3}\cdot\ldots\cdot p_{i_{n-1} i_n},$$

using the multiplication rule. In order to completely know this probability we need the distribution of the initial step, $X_1$.

# Path Probabilities and $n$-steps Transitions



- In many problems involving Markov chains we need the distribution of a future step depending on the current one.

**Definition 3**

$n$-step transitions probabilities *are*

$$r_{ij}^{(n)} = P\{X_{n+1} = s_j | X_1 = s_i\}.$$

- Because of the homogeneity, $r_{ij}^{(n)}$ is the probability that after $n$ steps the state becomes $s_j$, if the initial state was $s_i$ (at any initial step: $r_{ij}^{(n)} = P\{X_{n+k} = s_j | X_k = s_i\}$). All these probabilities can be computed using the following recursive equation.

## Path Probabilities and $n$-steps Transitions



### Proposition 2

*(Chapman-Kolmogorov equation) $n$-steps transitions probabilities are given by*

$$r_{ij}^{(n)} = \sum_{k=1}^{m} r_{ik}^{(n-1)} \cdot p_{kj}, \text{ for } n \geqslant 2, 1 \leqslant i,j \leqslant m, \text{ where } r_{ij}^{(1)} = p_{ij}.$$

proof: We use the conditional version of the total probability formula:

$$P\{X_{n+1} = s_j | X_1 = s_i\} = \sum_{k=1}^{m} P\{X_n = s_k | X_1 = s_i\} \cdot$$

$$\cdot P\{X_{n+1} = s_j | X_n = s_k, X_1 = s_i\} =$$

$$= \sum_{k=1}^{m} P\{X_n = s_k | X_1 = s_i\} \cdot P\{X_{n+1} = s_j | X_n = s_k\} = \sum_{k=1}^{m} r_{ik}^{(n-1)} \cdot p_{kj}.$$

■

# Path Probabilities and $n$-steps Transitions



- Matrix $r_{ij}^{(n)}$ (for a given $n$) is called the *$n$-step transition probability matrix*.
- From Chapman-Kolmogorov equations we get the following result (left as an exercise).

**Proposition 3**

*The $n$-step transition probability matrix is $P^n$, where $P$ is the transition probability matrix.*

These matrices are *stochastic matrices*: their elements are probabilities and the sum on every row is 1.

# Types of States



- In certain situation the probabilities $r_{ij}^{(n)}$ converge for $n \to +\infty$, no matter the initial state.

*Example (cont'd)* Consider the Markov chain having the following transition probability matrix

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}, P^2 = \begin{bmatrix} 0.76 & 0.24 \\ 0.72 & 0.28 \end{bmatrix}, P^3 = \begin{bmatrix} 0.752 & 0.248 \\ 0.744 & 0.256 \end{bmatrix},$$

$$\dots, P^{10} = \begin{bmatrix} 0.7500 & 0.2502 \\ 0.7508 & 0.2505 \end{bmatrix}.$$

We note that $n$-steps transition probability matrix converges to a constant matrix no matter the initial state (its columns are constant). ♣

# Types of States



*Example (cont'd)* For the chain with a bee and two spiders

$$P = \begin{bmatrix} 1.0 & 0 & 0 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0 & 0.3 & 0.4 & 0.3 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}, P^{20} = \begin{bmatrix} 1.0 & 0 & 0 & 0 \\ 0.669 & 0.0004 & 0.0004 & 0.329 \\ 0.329 & 0.0004 & 0.0004 & 0.669 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}. \clubsuit$$

In this case the limits depend on the initial state:

$$\lim_{n\to+\infty} r_{11}^{(n)} = 1, \lim_{n\to+\infty} r_{21}^{(n)} = 2/3,$$

$$\lim_{n\to+\infty} r_{31}^{(n)} = 1/3, \lim_{n\to+\infty} r_{41}^{(n)} = 0.$$

# Types of States



- The classification of states concerns the long-term frequency of visiting the states.

**Definition 4**

(i) *A* state $s_j$ *is* accessible *from another state,* $s_i$*, if there exists an integer* $n \geqslant 1$ *such that* $r_{ij}^{(n)} > 0$*; let* $A(s_i)$ *be the set of all accessible states from* $s_i$*.*

(ii) *A* state $s_i$ *is* recurrent *if, for every state* $s_j$ *which is accessible from* $s_i$*,* $s_i$ *is also accessible from* $s_j$*.*

(iii) *A* state *that is not recurrent is called* transient*.*

# Types of States



- State $s_j$ is accessible from $s_i$ if exists a sequence $s_{i_1}, s_{i_2}, \ldots, s_{i_{n-1}}$ such that
$$p_{ii_1}, p_{i_1 i_2}, \ldots, p_{i_{n-1}j} > 0,$$
(it exists a path from $s_i$ to $s_j$ having positive probability).
- State $s_i$ is recurrent if and only if, $\forall\, s_j \in A(s_i) \Rightarrow s_i \in A(s_j)$. If we start from a recurent state $s_i$, then the probability to visit again $s_i$ is positive (therefore we can visit $s_i$ an infinite number of times).
- Moreover, if $s_i$ is recurrent , then $A(s_i) = A(s_j)$, for every $s_j \in A(s_i)$: starting from $A(s_i)$ we stay forever in $A(s_i)$.

# Types of States



**Definition 5**

*If $s_i$ is a recurrent state, then $A(s_i)$ is called a* recurrent class.

- It can be easily proved (exercise): recurrent classes are equivalence classes of the following relation (which is an equivalence on the set of recurrent states): $s_i \sim s_j$ if $A(s_i) = A(s_j)$.

**Theorem 3.1**

*A Markov chain can be decomposed in some recurrent classes and (maybe) some transient states.*

- The following properties are left as exercises.

# Types of States



**Proposition 4**

*Consider a discrete homogenuous Markov chain with a finite number of states. Then*

(i) *A recurrent state is accessible only from every state in its class and possibly some other transient states but not from states belonging to another recurrent class.*

(ii) *A transient state it is not accessible from a recurrent state.*

(iii) *From a transient state is accessible at least a recurrent state.*

- Our recurrent class definition allows a recurrent class formed with a single state having a *loop* (a transition arc to itself) and not other outgoing arcs.

# Types of States



- The above results allow us reasoning on Markov chains and the visualization of their evolution:

(i) once we enter (or start from) a recurrent state, we cannot leave its recurrent class (each state in this class can be visited an infinite number of times) .

(ii) If the starting state is a transient one, then we will visit a number of transient states and, then, we will enter a recurent class without leaving it.

**Definition 6**

*A recurent class is called* periodic *if its states can be partitioned in $k \geqslant 2$ subsets $C_1, C_2, \ldots, C_k$, such that any transition take place only from a subset to another in this order (and circular):*

$$\forall\, s_i \in C_h, p_{ij} > 0 \Rightarrow s_j \in \begin{cases} C_1, & \text{if } h = k \\ C_{h+1}, & \text{otherwise} \end{cases} .$$

# Types of States



- if $s_i$ belongs to a periodic class, then for every $n \geqslant 1$ there exists a state $s_j$, such that $r_{ij}^{(n)} = 0$. In this way we get a criteria for non-periodicity:
- If there exist an $n \geqslant 1$ and an $s_i \in R$, such that $r_{ij}^{(n)} > 0$, for every $s_j \in R$, then $R$ is not a periodic class.
- For a Markov chain we are interested in its long-term behavior: that is, the probabilities $r_{ij}^{(n)}$, for large $n$.
- In this section, we give sufficient conditions for $r_{ij}^{(n)}$ to converge, no matter the initial state $s_i$.
- If there are two recurrent classes, then it is obvious that the above limits depend on the starting state (we cannot leave a recurrent class). As a consequence, we must suppose that our chain has only one recurrent class.

## The time to return to a specific state



We denote by $q_{ij}^t$ the probability that starting at state $s_i$, the first transition to state $s_j$ occurs at time $t$:

$$q_{ij}^t = P(X_t = s_j \text{ and, for } s = \overline{1, t-1}, X_s \neq s_j | X_0 = s_i).$$

**Definition 7**

*Let $Z_{ij}$ be the time to reach to state $s_j$ when starting at state $s_i$; its expectation, $h_{ij} = \mathbb{E}[Z_{ij}] = \sum_{t \geqslant 1} t q_{ij}^t$ is called the expected time to reach $s_j$ starting from $s_i$. ($Z_{ii}$ is the time to return to state $s_i$ and $h_{ii}$ is called the expected time to return to $s_i$.)*

- We can observe that, for a Markov chain with a finite number of states, the above expectations are finite.

## Long-term Behavior of Markov Chains



*Example.* Let us consider a Markov chain with only two states $\{s_1, s_2\}$, such that from $s_1$ we go to $s_2$ ($p_{12} = 1$) and from $s_2$ to $s_1$ ($p_{21} = 1$). Therefore:

$$r_{ii}^{(n)} = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases} . \clubsuit$$

- In this example $r_{ii}^{(n)}$ doesn't converge (it oscilates) because its only recurrent class is periodic. For convergence the chain must contain only one non-periodic class.
- The following theorem is the central result of today lecture.

# Long-term Behavior of Markov Chains



**Theorem 4.1**

*Consider a discrete homogenuous Markov chain with a finite number of states. If it contains only one recurrent class which is not periodic (and possibly some other transient states), then we can associate a steady state probability $\pi_j$ to each state $s_j$ such that:*

(i) $\lim\limits_{n\to+\infty} r_{ij}^{(n)} = \pi_j$, *for every $i$ and $j$.*

(ii) $(\pi_j)_{1\leqslant j\leqslant m}$ *are solutions to the linear system*

$$\begin{cases} \sum\limits_{k=1}^{m} \pi_k p_{kj} = \pi_j, \quad j=\overline{1,m} \\ \sum\limits_{k=1}^{m} \pi_k = 1 \end{cases}$$

(iii) $\pi_j = 0$, *if $s_j$ is transient and* $\pi_j = \dfrac{1}{h_{jj}} > 0$, *if $s_j$ is recurrent.*

# Long-term Behavior of Markov Chains



- Probabilities $\pi_j$ give a probability distribution on the states space: the stationary distribution - it is called like this because, if $X_1$ has such a distribution

$$P\{X_1 = s_j\} = \pi_j, \forall 1 \leqslant j \leqslant m, \text{ then}$$

$$P\{X_2 = s_j\} = \sum_{k=1}^{m} P\{X_1 = k\} p_{kj} = \sum_{k=1}^{m} \pi_k p_{kj} = \pi_j, \forall 1 \leqslant j \leqslant m$$

and $P\{X_n = s_j\} = \pi_j, \ \forall 1 \leqslant j \leqslant m, \forall n \geqslant 1.$

$$\sum_{k=1}^{m} \pi_k p_{kj} = \pi_j, j = \overline{1, m}$$

are balance equations (from Chapman-Kolmogorov equations).

$\sum_{k=1}^{m} \pi_k = 1$ is the normalization equation.

# Long-term Behavior of Markov Chains - Examples

*Example.* Consider a Markov chain, with two states and transition probabilities

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}.$$

*Solution:* Balance equations are

$$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} \text{ and } \pi_2 = \pi_1 p_{12} + \pi_2 p_{22},$$

therefore

$$\pi_1 = 0.8\pi_1 + 0.6\pi_2 \text{ and } \pi_2 = 0.2\pi_1 + 0.4\pi_2.$$

These two equations gives

$$\pi_1 = 3\pi_2.$$

Using the normalization equation $\pi_1 + \pi_2 = 1$, we get

$$\pi_1 = 0.75, \pi_2 = 0.25. \clubsuit$$

# Long-term Behavior of Markov Chains - Examples

*Example.* An absent-minded professor has two umbrellas that she (he) uses when commuting from home to office and back. If it rains and an umbrella is available in his location, she takes it. If it is not raining, she always forgets to take an umbrella. Suppose that it rains with probability $p$ each time she commutes, independently of other times. What is the steady-state probability that she gets wet during a commute?
*Solution:* We model this problem using a Markov chain with the following states: $s_i$: in its location the professor has $i$ umbrellas, $i = \overline{0,2}$. The transition probabilities are:

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}$$

# Long-term Behavior of Markov Chains - Examples



Note that our chain has only one recurrent class which is not periodic, therefore we can use the above theorem, the balance equations are

$$\pi_0 = (1-p)\pi_2, \pi_1 = (1-p)\pi_1 + p\pi_2 \text{ şi } \pi_2 = \pi_0 + p\pi_1.$$

By solving the system (adding the normalization equation) we get

$$\pi_0 = \frac{1-p}{3-p}, \pi_1 = \frac{1}{3-p}, \pi_2 = \frac{1}{3-p}. \clubsuit$$

# Random walks on undirected graphs

- A *random walk* on an undirected graph is a special type of Markov chain used in analyzing algorithms. Let $G = (V, E)$ be a (finite) undirected and connected graph.

**Definition 8**

*A* random walk on $G$ *is a Markov chain defined by the sequence of moves of a particle between vertices of* $G$ *(which are the states of the chain). If the particle is at vertex* $s_i$ *which has* $d_i = d_G(s_i)$ *neighbours, then the probability that the particle follows the edge* $s_i s_j$ *and moves towards* $s_j$ *is* $1/d_i$.

- $G$ being connected, the corresponding random walk will have exactly one recurrent class (why?).
- The associated transition probability digraph is obtained by orienting every edge in $G$ both ways. Let $\vec{G}$ be the transition digraph corresponding to the random walk on $G$.

## Random walks on undirected graphs

**Lemma 1.1**

*A random walk on an undirected connected graph $G$ is non-periodic if and only if $G$ is not bipartite.*

proof: A graph is bipartite if and only if it does not have odd cycles. Obviously, in $\vec{G}$ there is a path from every vertex to itself of length 2. If $G$ is bipartite then the walk has period 2. If $G$ is not bipartite then it contains an odd cycle and there exists a vertex which has an odd length path to itself. ■

- A random walk on an undirected, connected, and non-bipartite graph satisfies the conditions from Theorem 4.1.

**Theorem 1.1**

*Let $G$ be an undirected, connected, and non-bipartite graph. The steady state probabilities of a random walk on $G$ are:* $\pi_i = \dfrac{1}{h_{ii}} =$

## Random walks on undirected graphs



proof: Let $P = (p_{ij})$ be the transition probability matrix of the corresponding Markov chain. The stationary distribution obviously exists by Theorem 4.1 and we have no transient states in such a Markov chain. Let $(\pi_i)_{s_i \in V}$ be this distribution. The balance equations, $\pi = \pi P$, are equivalent with

$$\pi_j = \sum_{s_i \in N_G(s_j)} \frac{d_i}{2|E|} \frac{1}{d_i} = \frac{d_j}{2|E|}.$$

Now, from cited theorem we also get $\pi_j = \dfrac{1}{h_{jj}}$. (Since $\sum\limits_{s_i \in V} d_i = 2|E|$ it follows that $\sum\limits_{s_i \in V} \pi_i = 1$ which is the normalization equation.) ■

**Lemma 1.2**

*If $s_i s_j \in E$, then $h_{ij} < 2|E|$.*

## Random walks on undirected graphs

proof: First we compute $h_{ii}$ in a different way:

$$h_{ii} = \sum_{t \geqslant 1} t q_{ii}^{t} = \sum_{t \geqslant 1} t \sum_{s_j \in N_G(s_i)} p_{ij} q_{ji}^{t-1} = \frac{1}{d_i} \sum_{t \geqslant 1} \sum_{s_j \in N_G(s_i)} t q_{ji}^{t-1} =$$

$$= \frac{1}{d_i} \left[ \sum_{t \geqslant 1} \sum_{s_j \in N_G(s_i)} q_{ji}^{t-1} + \sum_{t \geqslant 1} \sum_{s_j \in N_G(s_i)} (t-1) q_{ji}^{t-1} \right] =$$

$$= \frac{1}{d_i} \left( \sum_{t \geqslant 1} \sum_{s_j \in N_G(s_i)} q_{ji}^{t-1} + \sum_{t \geqslant 1} \sum_{s_j \in N_G(s_i)} t q_{ji}^{t} \right) =$$

$$= \frac{1}{d_i} \sum_{s_j \in N_G(s_i)} \left( 1 + \sum_{t \geqslant 1} t q_{ji}^{t} \right) = \frac{1}{d_i} \sum_{s_j \in N_G(s_i)} (1 + h_{ji}).$$

We know that $h_{ii} = \frac{1}{\pi_i} = \frac{2|E|}{d_i}$, therefore $2|E| = \sum_{s_j \in N_G(s_i)} (1 + h_{ji})$, and,

from here we get $h_{ji} < 2|E|$, for every $s_j \in N_G(s_i)$.

# Random walks on undirected graphs



**Definition 9**

*The cover time of a graph $G$ ($cover(G)$) is the maximum over all vertices $s_i \in V$ of the expected time to visit all of the nodes in the graph by a random walk starting from $s_i$.*

**Lemma 1.3**

*The cover time of $G = (V, E)$ is at most $4mn$ ($n = |V|, m = |E|$).*

proof: We choose a spanning tree $T$ of $G$ and an Eulearian cycle which tour $\vec{T}$. Let $s_{i_1}, s_{i_2}, \ldots, s_{i_{2n}}$ be a sequence of vertices in the tour starting from a given vertex $s_{i_1}$ (for example the sequence of vertices passed through when performing a depth-first search).
The expected time to go through the vertices in the tour is an upper bound on the cover time:

$$cover(G) \leq \sum^{2n-1} h_{i_j, i_{j+1}} \leq 2m(2n-2) \leq 4mn.$$

# An $s$-$t$ Connectivity Algorithm



- Suppose that we are given an undirected graph $G = (V, E)$ ($|V| = n, |E| = m$) and two vertices $s, t \in V$.
- The problem is to determine a path connecting $s$ and $t$ if such a path exists.
- Solutions to this problem already exist: using a classic breadth-first search or depth-first search. The time complexity of these two algorithms is $\mathcal{O}(m + n)$, the required space being $\Omega(n)$ (you have to keep track of visited vertices).
- We present in this section a *randomized algorithm* that works with only $\mathcal{O}(\log n)$ bits of memory.
- Our algorithm will perform a random walk on $G$ for enough steps so that a path from $s$ to $t$ is likely to be found.

# An $s$-$t$ Connectivity Algorithm



start *a random walk from s*;
if(*the walk reaches t within* $4n^3$ *steps*)
  return *"there is a path"*;
return *"there is no path"*

- The algorithm keeps track of its current position using $\mathcal{O}(\log n)$ bits, and the number of steps taken (at most $4n^3$) which also takes $\mathcal{O}(\log n)$ bits.
- We suppose that the graph has no bipartite connected components in order to apply the former results in this section.

**Theorem 2.1**

*The above algorithm returns the correct answer with probability at least 1/2, and it errs only by returning that there is no path from s to t when there is such a path.*

## An $s$-$t$ Connectivity Algorithm



proof: If there is no path then the algorithm returns the correct answer. Suppose now that there is a path between $s$ and $t$. Let $X$ be the time to reach $t$ from $s$; the expected time to reach $t$ from $s$, $\mathbb{E}[X]$, is bounded from above by the cover time of their connected component, $G'$ (that contains $s$ and $t$), which is at most $4mn < 2n^3$. By Markov inequality the probability that the walk takes more than $4n^3$ steps is

$$P(X \geqslant 4n^3) < P(X \geqslant 8nm) \leqslant P(X \geqslant 2\mathbb{E}[X]) \leqslant \frac{\mathbb{E}[X]}{2\mathbb{E}[X]} = \frac{1}{2}.$$

Hence the algorithm errs with probability at most 1/2. ■

# Exercises for Seminar



- Markov Chains: 1, 2, 3, 5, 6, 9, 10, 16, 18, 19, 21
- Reserve: 11, 15, 20, 22

End

## Markov Chains - Exercises



1. A naive method to predict the weather is the following: tomorrow state is similar to that of today. This type of prediction is correct in 75% of cases. Suppose that we have only two types of weather: "sunny" and "rainy". Determine an appropriate Markov chain, its transition digraph, and the steady state probabilities.
2. The above method is modified for a sunny country: the transition probability from a rainy day to a sunny one is 0.5, the transition probability from a sunny day to a rainy one is 0.1. Rework the exercises in these conditions.
3. Draw the transition digraph for a Markov chain having a couple of recurrent classes and some transient states.
4. Draw the transition digraph for a Markov chain having one periodic recurrent class, and two transient states.
5. Exhibit the transition matrix of a Markov chain with period three.

## Markov Chains - Exercises



6. Draw the transition probability digraph for a Markov chain having two recurrent non-periodic classes and one transient state.

7. Define, on the set of recurrent states of a homogenuous Markov chain, the following relation: $s_i$ *communicate* with $s_j$ if $s_j$ is accessible from $s_i$ (we write $s_i \leftrightarrow s_j$). Prove that the communicating relation defines an equivalence relation, that is, the communicating relation is

1. *reflexive*: for any state $s_i$, $s_i \leftrightarrow s_i$;
2. *symmetric*: if $s_i \leftrightarrow s_j$, then $s_j \leftrightarrow s_i$, for every $i, j$;
3. *transitive*: if $s_i \leftrightarrow s_j$ and $s_j \leftrightarrow s_k$, then $s_i \leftrightarrow s_k$ for every $i, j, k$.

7 8*. An $n \times n$ matrix is called *double stochastic* if the sum of the entries in each row is 1 and the sum of the entries in each column is 1. Show that the uniform distribution is a stationary distribution for any Markov chain represented by such a matrix.

## Markov Chains - Exercises



9. A professor gives tests those level of difficulty is hard, medium, or easy. If he gives a hard test, his next test will be only medium or easy, with equal probability. However, if he gives a medium or easy test, there is a 0.5 probability that his next test will be of the same difficulty, and a 0.25 probability for each of the other two levels of difficulty. Define an appropriate Markov chain and find the steady-state probabilities.

10. A museum owns three paintings by Renoir, two by Cézanne, and one by Monet. It has room to display only one of these paintings. Therefore the painting on display is changed once a month. At that time, the painting on display is replaced by a randomly chosen one of the other five paintings. Let $s_1$ be "A Renoir is on display", $s_2$ be "A Cezanne is on display", and $s_3$ be "A Monet is on display." Find the transition matrix for the Markov chain just described.

## Markov Chains - Exercises



11. Rework the last exercise modified as follows: The next painting to be displayed is randomly chosen from among those paintings by different artists from the painting being replaced.

12. A hypermarket can have "big sales", "medium sales" or "low sales" days; in some days the hypermarket is closed for supplying. After a "big sales" day it is closed for supplying with probability 0.8 or the following day will be a "low sales" day. After a "medium sales" day it will follow a "big sales" day with probability 0.4 or "low sales" day with probability 0.6. After a "low sales" day or after a supplying day it will follow a "low sales" day with probability 0.3, or "medium sales" day with probability 0.3 or "big sales" day with probability 0.4.

(a) Define the Markov chain, draw the transition probability digraph and show that there is an unique recurrent non-periodic class.

(b) Find the steady state probabilities.

## Markov Chains - Exercises



13*. (Ehrenfest Diffusion) In a box we have a total of $n$ balls, some of them black, some white. At each time step, we either do nothing, which happens with probability $p \in (0,1)$ or we select a ball at random, so that each ball has probability $(1-p)/n > 0$ of being selected. In the latter case, we change the color of the selected ball (if white it becomes black, and vice versa), and the process is repeated indefinitely. What is the steady-state distribution of the number of white balls?

(*Hint: state $s_i$ may be: we have $i$ white balls in the box,* $0 \leqslant i \leqslant n$.)

14*. A superstitious professor works in a circular building with $m$ doors ($m$ is odd), and never uses the same door twice in a row. Instead he uses with probability $p$ (or probability $(1-p)$) the door that is adjacent in the clockwise direction (or the counterclockwise direction, respectively) to the door he used last. What is the probability that a given door will be used on some particular day far into the future?

(*Hint: state $s_i$ may be: the last used door is door $i$.*)

## Markov Chains - Exercises



15. A comptroller handles read-write requests from three processes directed toward a common memory device. Traffic conditions are such that un unlimited number of requests are always pending for each process. Upon completing a request from process $i$, the controller takes the next request from process $j$ with probability $p_{ij}$, where $P = (p_{ij})$ is the following matrix

$$\begin{bmatrix} 0.1 & 0.4 & 0.5 \\ 0.3 & 0.2 & 0.5 \\ 0.7 & 0.2 & 0.1 \end{bmatrix}$$

We can assume that read-write requests require a constant fixed time to service.

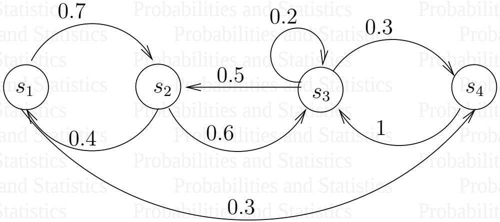
(a) Model this situation as a three-state Markov chain and draw the associated digraph.

(b) Compute the steady states probabilities, only if there exists only one recurrent, non-periodic class.

## Markov Chains - Exercises



16. A certain homogeneous discrete Markov chain has the following following transition digraph:

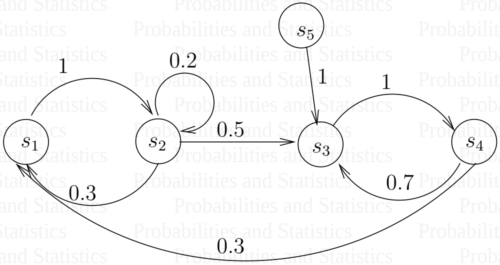



(a) Determine the transition matrix, the recurrent classes and the transient states.

(b) Compute the steady states probabilities, only if there exists only one recurrent, non-periodic class.

# Markov Chains - Exercises

17. The following digraph corresponds to an homogeneous discrete Markov chain:




(a) Determine the transition matrix, the recurrent classes and the transient states.

(b) If there exists only one recurrent, non-periodic class, then compute the steady states probabilities.

## Markov Chains - Exercises



18. Consider an homogenous Markov chain having the following transition probabilities matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.4 & 0 & 0.6 & 0 \\ 0 & 1 & 0 & 0 \\ 0.3 & 0 & 0 & 0.7 \end{bmatrix}$$

(a) Draw its transition digraph.

(b) Determine the recurrent classes and the transient states.

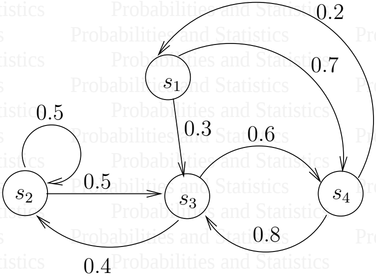
(c) Which of the recurrent classes are periodic?

19. Solve the same exercise for the following transition probabilities matrix

$$\begin{bmatrix} 0 & 0.2 & 0.6 & 0.2 \\ 0.2 & 0.6 & 0 & 0.2 \\ 0 & 0.4 & 0.6 & 0 \\ 0 & 0.3 & 0.7 & 0 \end{bmatrix}$$

## Markov Chains - Exercises

20. The following digraph corresponds to an homogeneous discrete Markov chain:




(a) Determine the transition matrix, the recurrent classes and the transient states.

(b) Compute the steady states probabilities, only if there exists only one recurrent, non-periodic class.

## Markov Chains - Exercises



21. A student's study habits are as follows: he studies one or two hours or he do nothing in every given day . If he studies two hours in a given day, then for the next day is 75% sure that he will do nothing and 25% sure he will study one hour. If he studies one hour in a given day, the next day there are 30% chances that he will do nothing, and 30% that he he will study again one hour. At last, if he doesn't study in a day, then in the next day there are 75% chances that he will study one hour, otherwise he will do nothing. A Markov chain can be defined if we identify a state with the number of hours dedicated to studying.

(a) Draw the transition digraph, determine the transition matrix, and indicate the recurrent classes and the transient states.

(b) If possible, compute the steady states probabilities.

# Markov Chains - Exercises

22. An art gallery accommodates each month a different exhibition: pictures, sculpture or photography exhibitions. After a picture exhibition it will follow, with same chances, a sculpture or a photography exhibition. After a photography exhibition, the chances to have a sculpture exhibition are twice the chances to have a picture exhibition. After a sculpture exhibition will follow every time a picture one. A Markov chain can be defined if we identify a state with the type of exhibition.

A Markov chain can be defined if we identify a state by the type of the exhibition.

(a) Draw the transition graph, determine the transition matrix, and indicate the recurrent classes and the transient states.

(b) If possible (justify why), describe the linear system those solutions are the steady states probabilities.

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