Probabilities and Statistics - Lecture 3

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Multiplication rule

Proposition 1

Let $A_1, A_2, \ldots, A_n$ random events, then

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) =$$

$$= P(A_1) \cdot P(A_2 | A_1) \cdot \ldots \cdot P(A_n | A_1 \cap A_2 \cap \ldots \cap A_{n-1}),$$

when $P(A_1 \cap A_2 \cap \ldots \cap A_{n-1}) > 0$.

proof: The right member is equal with

$$P(A_1) \cdot \frac{P(A_1 \cap A_2)}{P(A_1)} \cdot \frac{P(A_1 \cap A_2 \cap A_3)}{P(A_1 \cap A_2)} \cdot \ldots$$

$$\ldots \cdot \frac{P(A_1 \cap A_2 \cap \ldots \cap A_{n-1} \cap A_n)}{P(A_1 \cap A_2 \cap \ldots \cap A_{n-1})} = P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

(after some algebra). ■

Multiplication rule

Example. A box contains 5 white balls and 5 black balls. We withdraw 3 balls one by one without replacement.

(a) What is the probability that all the balls are white?

(b) What is the probability that two balls are white and one is black?

Solution:

(a) For the first question let A_i ="the i th ball is white" ($i = \overline{1,3}$), then

$$P(A_1) = \frac{1}{2},\ P(A_2|A_1) = \frac{4}{9},\ P(A_3|A_1 \cap A_2) = \frac{3}{8},$$

therefore

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) = \frac{1}{2} \cdot \frac{4}{9} \cdot \frac{3}{8} = \frac{1}{12}.$$

Multiplication rule

(b) For the second question the probability is

$$P\left(\left(\overline{A}_1 \cap A_2 \cap A_3\right) \cup \left(A_1 \cap \overline{A}_2 \cap A_3\right) \cup \left(A_1 \cap A_2 \cap \overline{A}_3\right)\right) =$$

$$P\left(\overline{A}_1 \cap A_2 \cap A_3\right) + P\left(A_1 \cap \overline{A}_2 \cap A_3\right) + P\left(A_1 \cap A_2 \cap \overline{A}_3\right).$$

Each of these probabilities can be computed using the multiplication rule. ♣

Multiplication rule

Example. A box contains 4 white balls and 6 black balls. we withdraw 3 balls one by one without replacement.

(a) What is the probability that all the balls are black?

(b) What is the probability that the first and the third balls are white while the second is black?

(c) What is the probability that two balls are black and one is white?

Solution:

(a) For the first question let $A_i =$"the i-th ball is black".

$$P(A_1) = \frac{6}{10},\ P(A_2|A_1) = \frac{5}{9},\ P(A_3|A_1 \cap A_2) = \frac{4}{8} \text{ and}$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) = \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} = \frac{1}{6}.$$

Multiplication rule

(b) For the second question the probability is

$$P\left(\overline{A}_1 \cap A_2 \cap \overline{A}_3\right) = P(\overline{A}_1) \cdot P(A_2 | \overline{A}_1) \cdot P(\overline{A}_3 | \overline{A}_1 \cap A_2),$$

and

$$P(\overline{A}_1) = \frac{4}{10}, P(A_2 | \overline{A}_1) = \frac{6}{9}, P(\overline{A}_3 | \overline{A}_1 \cap A_2) = \frac{3}{8}.$$

$$P\left(\overline{A}_1 \cap A_2 \cap \overline{A}_3\right) = \frac{4}{10} \cdot \frac{6}{9} \cdot \frac{3}{8} = \frac{1}{10}.$$

Multiplication rule

(c) For the third question the probability is

$$P\left((\overline{A}_1 \cap A_2 \cap A_3) \cup (A_1 \cap \overline{A}_2 \cap A_3) \cup (A_1 \cap A_2 \cap \overline{A}_3)\right) =$$

$$P\left(\overline{A}_1 \cap A_2 \cap A_3\right) + P\left(A_1 \cap \overline{A}_2 \cap A_3\right) + P\left(A_1 \cap A_2 \cap \overline{A}_3\right)$$

all these probabilities will be computed using the multiplication rule. As an example

$$P(A_1 \cap \overline{A}_2 \cap A_3) = P(A_1) \cdot P(\overline{A}_2 | A_1) \cdot P(A_3 | A_1 \cap \overline{A}_2) = \frac{6}{10} \cdot \frac{4}{9} \cdot \frac{5}{8} = \frac{1}{6}. \clubsuit$$

Hypergeometric Schema

- This schema applies for the following framework: in a box we have n_1 white balls and n_2 black balls, $n = n_1 + n_2$. We withdraw from this box, without replacement, k balls.

Proposition 2

Let $k_1, k_2 \in \mathbb{N}$, such that $k_1 \leqslant n_1$, $k_2 \leqslant n_2$ and $k = k_1 + k_2$. The probability that from those k balls, exactly k_1 are white and k_2 are black is

$$\frac{\binom{n_1}{k_1} \cdot \binom{n_2}{k_2}}{\binom{n}{k}}.$$

Hypergeometric Schema

- More general: in a box we have n_1 balls of color c_1, n_2 balls of color c_2, ..., n_p balls of color c_p (where $n = n_1 + n_2 + \cdots + n_p$). We withdraw from this box, without replacement, k balls.

Proposition 3

Let $k_1, k_2, \ldots, k_p \in \mathbb{N}$, such that $k_i \leqslant n_i$, $1 \leqslant i \leqslant p$ and $k = k_1 + k_2 + \cdots + k_p$. The probability that from those k balls, exactly k_1 are of color c_1, k_2 are of color c_2, ..., and k_p are of color c_p is

$$\frac{\binom{n_1}{k_1} \cdot \binom{n_2}{k_2} \cdot \ldots \cdot \binom{n_p}{k_p}}{\binom{n}{k}}.$$

Hypergeometric Schema

proof: Obviously the total number of combinations is $\binom{n_1+n_2}{k_1+k_2} = \binom{n}{k}$. Among these there are exactly $\binom{n_1}{k_1}$ possibilities to get exactly k_1 white balls, and for each of these possibilities there are another $\binom{n-n_1}{k-k_1} = \binom{n_2}{k_2}$ possibilities for getting $k - k_1 = k_2$ black balls.

■

Hypergeometric Schema

Example. In a box we have 4 white balls, 3 red balls, 5 black balls, and 4 blue balls. We withdraw from this box, without replacement, 7 balls.

(a) What is the probability that 2 of the balls are white, 3 are black, and 2 blue?

(b) What is the probability that exactly 4 of the balls are black?

Solution:

$$(a)\ \frac{\binom{4}{2}\cdot\binom{5}{3}\cdot\binom{3}{0}\cdot\binom{4}{2}}{\binom{16}{7}},$$

$$(b)\ \frac{\binom{5}{4}\cdot\binom{11}{3}}{\binom{16}{7}}.\clubsuit$$

Poisson Schema

- The framework of this schema is the following: we consider a random experience and n independent random events (related to this experience): $A_1, A_2, \ldots, A_n$ with known probabilities:

$$P(A_1) = p_1, P(A_2) = p_2, \ldots, P(A_n) = p_n.$$

Proposition 4

Let $k \in \mathbb{N}$, $k \leqslant n$. The probability that exactly k events occur (from those n) equals the coefficient of x^k from the polynomial

$$(p_1 x + q_1) \cdot (p_2 x + q_2) \cdot \ldots \cdot (p_n x + q_n),$$

where $q_i = P(\overline{A_i}) = 1 - p_i$, $i = \overline{1, n}$.

Poisson Schema

proof: Let $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ be the k events that occur ($1 \leqslant i_j \leqslant n$). In the same time will occur the events $\overline{A}_i$, with $i \in \{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_k\}$. These are independent events; the probability of their intersection is

$$\left(\prod_{j=1}^{k} p_{i_j}\right) \cdot \left(\prod_{i \notin \{i_1, i_2, \ldots i_k\}} q_i\right);$$ we add all these probabilities:

$$\sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} \left(\prod_{j=1}^{k} p_{i_j}\right) \cdot \left(\prod_{i \notin \{i_1, i_2, \ldots i_k\}} q_i\right),$$

but this sum may be obtained in other way: by adding up all the products of the following form: for k factors of our polynomial we choose the coefficient of x, and for the rest we choose the free term (from the corresponding binomial). The result is the coefficient of x^k.

■

Binomial Schema

- We consider a random experience and a random event A with probability $P(A) = p$. We independently perform the experience n times.

Proposition 5

The probability that event A occurs exactly k times $(0 \leqslant k \leqslant n)$ is $p^k(1-p)^{n-k}\binom{n}{k}$.

Binomial Schema

proof: Let the random experience be $\mathcal{E}$. We can define a new experiment $\mathcal{E}'$ like this: we independently perform $\mathcal{E}$, n times. We associate to this experiment the following random events: A_i - an event that occurs only if at the i-th performing of $\mathcal{E}$ the event A occurs.
We are, in this way, in the framework of Poisson schema, with n independent events $A_1, A_2, \ldots, A_n$ each having the same probability p. The required probability is the coeffcient of x^k in the development of the binomial $[px + (1-p)]^n$. ■

Binomial Schema

Example. We roll two dice 10 times. What is the probability that the product of the two values is 12 exacly 6 times?
Solution: Let $A =$"the product is 12" (at one roll),

$$A = \{(2,6),(3,4),(4,3),(6,2)\},$$

therefore $P(A) = 4/36 = 1/9$. The probability that A occurs exactly 6 times is

$$\frac{1}{9^6} \cdot \frac{8^4}{9^4} \cdot \binom{10}{6} . \clubsuit$$

Geometric Schema

- Framework: we independently perform a random experiment until a designated event, A, occurs ($P(A) = p$).

Proposition 6

The probability that event A occurs first time at the n-th performing of the experience ($n \geqslant 1$) is $p(1-p)^{n-1}$.

proof: Let the random experience be $\mathcal{E}$. We can define a new random experiment $\mathcal{E}'$ like follows: we independently perform $\mathcal{E}$, n times. We associate to this experiment the following random events: A_i - an event that occurs only if at the i-th performing (from $\mathcal{E}'$) of $\mathcal{E}$ the event A occurs.

The probability that event A occurs first at the n-th performing of the experience is

$$P(\overline{A}_1 \cap \overline{A}_2 \cap \ldots \cap \overline{A}_{n-1} \cap A_n) = p(1-p)^{n-1}.$$

Geometric Schema

Example. An urn contains 3 red balls, 2 black and 4 blue. We withdraw a ball and then we put it in the same urn until we get a blue or a red ball. What is the probability that only at the fourth withdrawal we get a red or a blue ball?

Solution: Let A ="the withdrawn ball is red or blue", $P(A) = 7/9$. The probability that event A occurs only at the fourth withdrawal:

$$\frac{7}{9} \cdot \frac{2^3}{9^3}. \clubsuit$$

Discrete Random Variable - Introduction

- After we perform a random experiment we are often interested in computing the value of (or to measure) a function which is associated with the possible oucomes: sum/product of two dice, the number of heads when tossing a coin etc.
- This it is possible because in many cases the outcomes are quantitative ones (real or integer numbers).
- The numerical result of the measurement of an outcome it is called a *random variable* - because of its unknown variation.
- Informally *random variable* is a function associating to each random event a number - which may be the result of a measurement.

Distribution of a Discrete Random Variable

Definition 1

Let $\mathcal{E}$ a random experience, Ω the sample space, a real random variable *is a function $X : \Omega \to \mathbb{R}$, such that for every interval $J \subseteq \mathbb{R}$, $X^{-1}(J)$ is a random event.*

Definition 2

A random variable *is called* discrete*, if its image is a countably (finite or infinite) set, that is $|X(\Omega)| \leqslant \aleph_0$. Otherwise it is a* continuous random variable.

- If the sample space is a discrete set (i.e., $|\Omega| \leqslant \aleph_0$), then an associated random variable can only be discrete.

Distribution of a Discrete Random Variable

- If $X(\Omega) = \{x_1, x_2, \ldots, x_n, \ldots\}$, then the set of all pairs (x_i, p_i) is the *distribution* or the *repartition* of the discrete random variable X, denoted by the following table

$$X : \begin{pmatrix} x_1 & x_2 & \ldots & x_n & \ldots \\ p_1 & p_2 & \ldots & p_n & \ldots \end{pmatrix} \tag{1}$$

- We use the notations $P\{X = x_i\} = P(X = x_i) = p_i$. Obviously, the sum (which may be a series) of the probabilities is 1:

$$\sum_i p_i = 1, 0 < p_i \leqslant 1, \forall\, i$$

Distribution of a Discrete Random Variable

Definition 3

Let $X : \Omega \to \mathbb{R}$ *discrete random variable*

(i) *The* probability mass function *of* X *is* $f_X : X(\Omega) \to [0, 1]$, *defined by* $f(x_i) = p_i = P\{X = x_i\}, \forall\, x_i \in X(\Omega)$.

(ii) *The* distribution function *of* X *(discrete or continuous) is* $F_X : \mathbb{R} \to [0, 1]$, *defined by*

$$F(a) = P\{X \leqslant a\}$$

Distribution of a discrete random variable

- Each of the above functions completely defines the random variable: the information related to the most refined partition (by X) of the sample space is contained by each of these functions.
- It is the case that two different random variables may have the same distribution.
- A random variable X is often called simply a distribution or repartition, covering in this way the entire family of random variables which have the same distribution function as X.

Proposition 7

Let $F_X : \mathbb{R} \to [0,1]$ the distribution function of X.

(i) *F_X is a non-decreasing function: $F_X(a) \leqslant F_X(b)$, for all $a < b$.*

(ii) $\lim\limits_{a \to +\infty} F_X(a) = 1$ *and* $\lim\limits_{a \to -\infty} F_X(a) = 0$.

Distribution of a discrete random variable

- As already noted, the probabilities associated with the distribution of X can be determined using its function of distribution.

Proposition 8

If X is a random variable, then

$$P\{a < X \leqslant b\} = F_X(b) - F_X(a).$$

- Discrete random variable are given mostly by the probability mass function, while the continuous ones are given by the distribution function (or, as we will see, by its probability density function).

Distribution of a discrete random variable - examples

Example. Suppose that a discrete random variable X has four values x_1, x_2, x_3, x_4, cu $x_1 < x_2 < x_3 < x_4$ and corresponding probabilities

$$P\{X = x_1\} = 0.2, P\{X = x_2\} = 0.3,$$

$$P\{X = x_3\} = 0.1, P\{X = x_4\} = 0.4,$$

the distribution function is

$$F_X(a) = \begin{cases} 0, & a < x_1 \\ 0.2, & a \in [x_1, x_2) \\ 0.5, & a \in [x_2, x_3) \\ 0.6, & a \in [x_3, x_4) \\ 1, & a \geqslant x_4 \end{cases} \quad \clubsuit$$

Distribution of a discrete random variable - examples

Example. In a box we have 4 white, 3 red, and 3 blue balls. We withdraw, without replacement, 2 balls. For each white ball we win 1$ and for each blue ball we loose 1$. Let X be the win; determine the distribution of X.

Solution: The possible values of X are $\{\pm 1, 0, \pm 2\}$; we determine the probability mass function (using hypergeometric schema)

$$f_X(-2) = P\{X = -2\} = \frac{\binom{3}{2}}{\binom{10}{2}} = \frac{3}{45} = \frac{1}{15},$$

$$f_X(-1) = P\{X = -1\} = \frac{\binom{3}{1} \cdot \binom{3}{1}}{\binom{10}{2}} = \frac{9}{45} = \frac{3}{15},$$

Distribution of a discrete random variable - examples

$$f_X(0) = P\{X = 0\} = \frac{\binom{3}{2} + \binom{4}{1} \cdot \binom{3}{1}}{\binom{10}{2}} = \frac{15}{45} = \frac{1}{3},$$

$$f_X(1) = P\{X = 1\} = \frac{\binom{4}{1} \cdot \binom{3}{1}}{\binom{10}{2}} = \frac{12}{45} = \frac{4}{15},$$

$$f_X(2) = P\{X = 2\} = \frac{\binom{4}{2}}{\binom{10}{2}} = \frac{6}{45} = \frac{2}{15}.$$

Distribution of a discrete random variable - examples

The distribution of X is

$$X : \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ \frac{1}{15} & \frac{3}{15} & \frac{5}{15} & \frac{4}{15} & \frac{2}{15} \end{pmatrix}. \clubsuit$$

Exercises for seminar

- Multiplication Formula: I.2, I.3, I.4.
- Probabilistic Schemata: II.3, II.4, III.4, III.5, III.8, IV.1, IV.4, IV.7.
- Discrete Random Variables: V.2.
- Reserve: II.2, III.2, III.9, IV.2, V.3.

End

Exercises - Multiplication Formula

I.1. In a box we have 6 white balls, 4 blue, and 2 red. We withdraw three balls without replacement. What is the probability that

(a) the first ball is white, and the other two are blue?

(b) a ball is blue, and two are red?

I.2. In a box we have 3 white balls, 2 black, and 4 red. We withdraw three balls without replacement. What is the probability that

(a) all balls are red?

(b) the first ball is black and the last two are red?

Exercises - Multiplication Formula

I.3. We have three urns. One contains 3 white balls and 1 black, the second contains 4 white balls and 5 black, and the last contains 1 white balls and 4 black. From the first urn a ball is withdrawn and transferred to the second, then from the second urn a ball is withdrawn and transferred to the third; at last a ball is withdrawn from the third urn. What is the probability that

(a) all three balls are white?
(b) two balls are black and one is white?
(c) at least a ball is white?

I.4. A box contains five white balls and seven black balls. Anytime we withdraw a ball from the box we replace it with two new balls of the opposite colour. We make three such withdrawals. Find the probability that

(a) the first two withdrawn balls are of different colours.
(b) the first ball is white and the following two are black.

Exercises - Multiplication Formula

I.5. Two balls are coloured with blue or green and then introduced in a box. A ball is green coloured with probability 1/3.

(a) If we know that blue colour was used (that is, at least a ball is blue coloured), what is the probability that both balls are blue?

(b) The box is overthrown and a blue ball falls. What is the probability that the remaining ball is green?

Exercises - Hypergeometric Schema

II.1. From a deck of (52) cards we withdraw four cards.

(a) What is the probability that exactly two of them are red?

(b) What is the probability that exactly one of them is black?

II.2. In a bag there are 4 black balls and 5 white balls. We withdraw 4 balls from the bag. What is the probability that

(a) two balls are white and two are black?

(b) all balls are black?

(c) a ball is white and three are black?

Exercises - Hypergeometric Schema

II.3. In a box there are 3 red balls, 4 blue, and 5 green; four balls are withdrawn from the box. What is the probability that

(a) two balls are red, one is blue, and one is green?

(b) a ball is red, one is blue, and two are green?

(c) three balls are blue and one is green?

II.4. From a deck of (52) cards we withdraw seven cards.

(a) What is the probability that exactly two of them are odd numbers and three are figures?

(b) What is the probability that exactly three of them are numbers numbers of diamonds and two of them are figures of clubs? (Ace is not a figure nor a number.)

Exercises - Binomial Schema

III.1. We flip one hundred coins.

(a) What is the probability that we get fifty tails?

(b) But the probability that we get at least fifty tails?

III.2. HDD Ltd. company produces hard-disks that have malfunctions with probability of 0.05. The company sells its hd's in bunches of ten and guarantees that such a package contains at most one defective hd, otherwise the entire package will be replaced. What is the probability for a given package to be replaced?

III.3. An archer athlete hit the target with a probability of 0.5; he shoots ten times. What is the probability that

a) the target is hit exactly five times?

a) the target is hit at least five times?

Exercises - Binomial Schema

III.4. We toss two dice ten times. What is the probability that

a) exactly five times the sum is greater or equal to 6, but the second die is different from 4?

b) at most eight times the sum is a prime number?

III.5. Facing an equal opponent, what is more probable to win: two games out of four or three out of six?

III.6. A communication channel randomly and independently transmits bits (0 with probability 0.25). The bits are received in pairs. If seven pairs of bits are transmitted, what is the probability that

a) exactly five times the received pair is an $0-1$?

b) at most six times we receive an $1-1$?

Exercises - Binomial Schema

III.7. A pair of dice is rolled six times. One die has number 3 on all faces, the other one being a fair one. What is the probability that

a) exactly four times the product of the dice is an even number?

b) at least four times the sum is a prime number?

III.8. A pair of coins is flipped five times. One of the coins has the probability of getting head equal with $1/3$, the other one being a fair one. What is the probability that

a) exactly three times the coins have different faces?

b) at least four times we get head on both coins?

Exercises - Binomial Schema

III.9. A pair of dice is tossed six times. What is the probability that

a) exactly three times the minimum of the two values we get is 3?

b) at least four times the sum of the values is greater than 6?

III.10*. In the framework of the binomial schema: an experience is independently performed n times, and we have in mind a random event A, $P(A) = p$, related with this experience; what number of occurrences of A is more probable?

Exercises - Geometric Schema

IV.1. From a deck of 52 cards we randomly withdraw a card (which is then returned to the deck) until we get an ace. What is the probability that only at the fifth withdrawal we get an ace?

IV.2. We toss two dice until we get a sum at least 8. What is the probability that

(a) this happens only at the third toss?

(b) this happens in one of the first two tosses?

IV.3*. Two players successively roll two dice. The winner is the player get first a total at most 9. What are the chances of winning for the player which starts the game?

Exercises - Geometric Schema

IV.4. We toss two coins many times. What is the probability that

(a) the first time we get two heads is at the fourth toss?

(b) the first time we get exactly one head is at the fourth toss?

IV.5. A card from a regular deck is withdrawn and then is returned to the deck. We perform this experience many times. What is the probability that

(a) the first time we get a face clubs is at the third withdrawal?

(b) in the first four withdrawals we get no diamonds?

IV.6. A communication channel randomly and independently transmits bits (0 with probability 0.4). The bits are received in pairs. What is the probability that

(a) the first time we get an $1-0$ occurs with the fourth pair?

(b) the first time we get an $1-1$ occurs with the third pair?

Exercises - Geometric Schema

IV.7. A pair of dice is rolled many times. One of the die has number 2 on all faces, the other one being a fair one. What is the probability that

a) the first time the product is a prime number occurs at the fourth roll?

b) the first time we get a double occurs at the third roll?

IV.8. We toss a pair of coins many times. One of the coins has the probability of getting head equal with 1/4, the other being a fair one. What is the probability that

(a) the first time we get head on both coins occurs at the fifth toss?

(b) no earlier than the third toss we get tail on both coins?

Exercises - Geometric Schema

IV.9. A pair of dice is tossed many times. What is the probability that

(a) the first time the maximum of the two values we get is 5 occurs at the fourth toss?

(b) the first time the sum of the values is less than 6 occurs at the fifth toss?

Exercises - Distribution of Discrete Random Variable

V.1*. A biased coin has the probability of head to occur at one flip equal with 2/3. The coin is flipped four times. Let X be the random variable equal with the maximum number of consective head. Compute the distribution of X.

V.2. Two balls are randomly chosen from a bag which contains 8 white balls, 4 black, and 2 yellow. Suppose that a black ball worth 2$, and a white one 1$. Let X be the total win; determine the distribution of X.

V.3. Let X be the difference between the number of occurrences of the head and the number of occurrences of the tail in three tosses of a coin. Determine the distribution and the repartition function of X

V.4. A die is tossed two times. Let X_1 and X_2 be the two results.

(a) Compute the distribution of $X = \min\{X_1, X_2\}$.

(b) Compute the distribution of $Y = \max\{X_1 + X_2, X_1 \cdot X_2\}$.

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