Seminar 8
1. A Column Generation approach for graph coloring (A. Mehrotra, M. A. Trick)

An ILP model

Let $G = (V, E)$ be a graph with n vertices and m edges. A p-coloring is a function $c : V \to \{1, 2, \ldots, p\}$ such $c(u) \neq c(v)$ for any edge $uv \in E$. A coloring class with respect to a coloring c is $c^{-1}(i)$ for a color i; the coloring classes are stable sets.

The minimum graph coloring problem (CP) : find the minimum p such that G has a p-coloring.

Let $\mathcal{S}$ be the set of all maximal stable sets of G; CP can be formulated like follows (see [6])

$$(CP)\begin{cases} \min & \sum\limits_{S\in\mathcal{S}} x_S \\ \text{s. t.} & \sum\limits_{S\in\mathcal{S},u\in S} x_S \geqslant 1, \forall u \in V, \\ & x_S \in \{0,1\}, \forall S \in \mathcal{S} \end{cases}.$$

The relaxation of (CP) is

$$(CPr)\begin{cases} \min & \sum\limits_{S\in\mathcal{S}} x_S \\ \text{s. t.} & \sum\limits_{S\in\mathcal{S},u\in S} x_S \geqslant 1, \forall u \in V, \\ & x_S \geqslant 0, \forall S \in \mathcal{S}. \end{cases}$$

(Remark that for an optimal solution of (CPr) $x_S \leqslant 1$, for any $S \in \mathcal{S}$.)

The corresponding dual is

$$(CPd)\begin{cases} \max & \sum\limits_{u\in V} y_u \\ \text{s. t.} & \sum\limits_{u\in S} y_u \leqslant 1, \forall S \in \mathcal{S}, \\ & y_u \geqslant 0, \forall u \in V. \end{cases}$$

The subproblem

A new variable x_S corresponding to a stable set S has a negative reduced cost if and only if for this stable set the corresponding constrained from the dual is violated:

$$\sum_{u\in S} y_u > 1.$$

Therefore, the subproblem is

$$(SubPr)\begin{cases} \max & \sum\limits_{u\in V} y_u z_u \\ \text{s. t.} & z_u + z_v \leqslant 1, \forall uv \in E, \\ & z_u \in \{0,1\}, \forall u \in V, \end{cases}$$

where $(y_u)_{u\in V}$ is an optimal solution to (CPd), provided that this maximum is > 1. This problem is the Maximum Weighted Stable Set (MWSS) and can be solved using an ILP solver or by using an algorithm like those given in [3].

Branch & Price algorithm

Initially, we can use a small family of maximal stable subsets (at least one for each vertex in the graph). We can add also the coloring classes of a coloring obtained by greedy algorithm - thus providing an upper bound for the chromatic number.

Let $\mathcal{S}'$ be the set of all maximal stable sets used for defining the problem in the current node of the Branch & Bound tree after this problem was solved to optimality using Column Generation (see [1], [5]). The corresponding restricted master problem (RMP) is

$$(RMP)\begin{cases} \min & \sum\limits_{S\in\mathcal{S}'} x_S \\ \text{s. t.} & \sum\limits_{S\in\mathcal{S}', u\in S} x_S \geqslant 1, \forall u \in V, \\ & x_S \geqslant 0, \forall S \in \mathcal{S}'. \end{cases}$$

Suppose that the optimal solution to this problem is fractional: let $x_{S_1} \in (0,1)$ and $u \in S_1$, since

$$\sum_{S\in\mathcal{S}', u\in S} x_S \geqslant 1,$$

there exists another stable set $S_2 \neq S_1$, $u \in S_2$ and we can choose a vertex $v \in S_2 \setminus S_1$.The **branching rule** is:

- in one (the left) child of the current node u and v will have the same color (It's like we contract the pair (u, v)) and
- in the other (the right) child u and v will have different colors (It's like we add an edge between u and v).

By using this branching rule we must enforce the following changes of $\mathcal{S}'$:

- in one (left) child of the current node we keep from $\mathcal{S}'$ only the stable sets containing both u and v ore none of them and
- in the other (right) child we must remove from $\mathcal{S}'$ the stable sets containing both u and v.

Let *Same* be the set of pairs of vertices which must have the same colors and $Different$ be the set of pairs of vertices which must have different colors, respectively in a given node of the Branch & Bound tree (initially, these sets are empty), then the subproblem must be changed like follows:

$$(SubPr')\begin{cases} \max & \sum\limits_{u\in V} y_u z_u \\ \text{s. t.} & z_u + z_v \leqslant 1, \forall uv \in E, \\ & z_u - z_v = 0, \forall (u,v) \in Same, \\ & z_u + z_v \leqslant 1, \forall (u,v) \in Different, \\ & z_u \in \{0,1\}, \forall u \in V. \end{cases}$$

2. Multiple-Depot Vehicle Scheduling Problem

We introduce an heuristic for the Multiple-Depot Vehicle Scheduling Problem (MDVSP) based on the interpretation of graph theoretic properties of a fractional solution.

Keywords: MDVSP, linear programming, cutting plane heuristic, relaxation.

LP model and heuristic

We use the classical model of [2]:

$$\text{(1)} \qquad min \sum_{i=0}^{m+n-1} \sum_{j=0}^{m+n-1} c_{ij} x_{ij}$$

$$\text{(2)} \qquad \sum_{i=0}^{m+n-1} x_{ij} = r_j, 0 \leqslant j \leqslant m+n-1$$

$$\text{(3)} \qquad \sum_{j=0}^{m+n-1} x_{ij} = r_i, 0 \leqslant i \leqslant m+n-1$$

$$\text{(4)} \qquad \sum_{ij \in E(D)} x_{ij} \leqslant |E(D)| - 1, D \in \mathcal{D}$$

$$\text{(5)} \qquad x_{ij} \in \mathbb{Z}_+, 0 \leqslant i, j \leqslant m+n-1$$

where $\mathcal{D}$ is the set of the (inclusionwise minimal) infeasible paths, that is the paths connecting two different depots. We relax this integer problem by replacing integrality constraints (5) with

$$\text{(5')} \qquad x_{ij} \geqslant 0, 0 \leqslant i, j \leqslant m+n-1$$

The set $\mathcal{D}$ is very large, even for small number of depots and an exact solution to problem (1) - (4), (5') can be obtained by using enumerative techniques. Our heuristic consist in replacing $\mathcal{D}$ by a smaller set of infeasible subtours (paths that links two different depots) by adding one by one constraints of type (4) and reoptimizing until the new problem has the same optimum as (1) - (4), (5').

Thus, at a certain step during the algorithm we have a particular set of infeasible subtours $\mathcal{D}'$ and (4) is replaced in the current problem by

$$\text{(4')} \qquad \sum_{ij \in E(D)} x_{ij} \leqslant |E(D)| - 1, D \in \mathcal{D}'$$

Consider, $\mathbf{x}^* = (x^*_{ij})$, a solution to problem (1) - (3), (4'), (5'), and define a weight on the edges of the underlying digraph: $\alpha_{ij} = 1 - x^*_{ij}$, for all arcs ij. (4) is equivalent with

$$\text{(4'')} \qquad \alpha(D) \geqslant 1, D \in \mathcal{D}$$

since

$$\sum_{ij \in E(D)} x_{ij} \leqslant |E(D)| - 1 \Leftrightarrow \sum_{ij \in E(D)} (1 - x_{ij}) \geqslant 1 \Leftrightarrow \sum_{ij \in E(D)} \alpha_{ij} \geqslant 1, \forall D \in \mathcal{D}.$$

Hence, $\mathbf{x}^*$ is an optimum solution to (1) - (4), (5') if and only if the underlying digraph doesn't contain paths between different depots of subunitary weight. We will test this by using algorithms like Floyd-Warshall or Bellman-Ford-Moore.

Hence the first step in our heuristic is to relax the problem (1) - (4), (5') to (1) - (3), (4'), (5') for a certain known set of infeasible paths $\mathcal{D}'$ such that the two problems have the same optimum. The process of building problem (1) - (3), (4') is given below.

$\mathcal{D}' \leftarrow \varnothing$;
solve problem (1) - (3), (4') and let $\mathbf{x}^*$ be an optimum solution;
while (there exists a path D with $\alpha(D) < 1$) **do**
  add D to $\mathcal{D}'$;
  solve problem (1) - (3), (4') and let $\mathbf{x}^*$ be an optimum solution;
end while
return $\mathbf{x}^*$.

The aim of the above procedure is to build a problem that has a larger (but known) set of feasible solutions but the same optimum with (1) - (4), (5')

Building a feasible solution

The next step is to solve the problem (1) - (3), (4'), (5) which is an integer linear programming problem. A solution to this problem may not be feasible for (1) - (5), thus a process of clearing the infeasible tours follows.

We describe now the heuristic for transforming the infeasible paths associated to an optimal solution $\mathbf{x}^*$ of problem (1) - (3), (4'), (5').

First we define an auxiliary digraph $H = (W, A)$, where W is the set of depots and $ij \in A$ if and only if there is an infeasible subtour between the depots i and j; we add also a weight on this arc w_{ij} which represents the number of such subtours. By inspecting $\mathbf{x}^*$ we extract the infeasible tours and memorize them, build H and the weight w. Than, each infeasible path (described by its arcs):

$$it_1, t_1t_2, \ldots, t_{p-1}t_p, t_pj,\ i, j \in W,$$

is replaced de by

$$it_1, t_1t_2, \ldots, t_{p-1}t_p, t_pi$$

if $c_{t_pi} - c_{t_pi} < c_{jt_1} - c_{it_1}$ or by

$$jt_1, t_1t_2, \ldots, t_{p-1}t_p, t_pj,$$

otherwise.

2. Benders Decomposition

This is a technique for solving MILP. Suppose we have to solve the following MILP problem:

$$\begin{array}{rl} \text{minimize} & \mathbf{c}^T\mathbf{x} + \mathbf{q}^T\mathbf{y}, \\ \text{subject to} & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \geqslant \mathbf{b}, \\ & \mathbf{y} \in Y, \\ & \mathbf{x} \geqslant \mathbf{0}, \end{array} \tag{6}$$

where Y could be the set of integer solutions to another LP, e. g.

$$Y = \{\mathbf{y} \in \mathbb{Z}^n_+ \ :\ \mathbf{C}\mathbf{y} = \mathbf{e}\}.$$

If $\mathbf{y}_0$ is fixed, the problem to solve becomes

$$\begin{array}{rl} \text{minimize} & \mathbf{c}^T\mathbf{x}, \\ \text{subject to} & \mathbf{A}\mathbf{x} \geqslant \mathbf{b} - \mathbf{B}\mathbf{y}_0, \\ & \mathbf{x} \geqslant \mathbf{0}. \end{array} \tag{7}$$

Problem (6) is equivalent with

$$\min_{\mathbf{y}\in Y}\left[\mathbf{q}^T\mathbf{y}+\min_{\mathbf{x}\geqslant\mathbf{0}}\{\mathbf{c}^T\mathbf{x}\;:\;\mathbf{A}\mathbf{x}\geqslant\mathbf{b}-\mathbf{B}\mathbf{y}\}\right]. \tag{8}$$

The dual of the inner problem is

$$\begin{array}{rl} \text{maximize} & (\mathbf{b}-\mathbf{B}\mathbf{y}_0)^T\mathbf{v},\\ \text{subject to} & \mathbf{A}^T\mathbf{v}\leqslant\mathbf{c},\\ & \mathbf{v}\geqslant\mathbf{0}, \end{array} \tag{9}$$

this is the subproblem.

The master problem is

$$\begin{array}{rl} \text{minimize} & z,\\ \text{subject to} & \text{ some constraints },\\ & \mathbf{y}\in Y. \end{array} \tag{10}$$

The Benders' decomposition algorithm

let $y_0\in Y$ an initial feasible solution;
$l\leftarrow-\infty$;
$u\leftarrow\infty$;
while $(u-l>\epsilon)$ **do**
  solve the subproblem:
  $\max_{\mathbf{v}}\{\mathbf{q}^T\mathbf{y}_0+(\mathbf{b}-\mathbf{B}\mathbf{y}_0)^T\mathbf{v}\;:\;\mathbf{A}^T\mathbf{v}\leqslant\mathbf{c},\mathbf{v}\geqslant\mathbf{0}\}$;
  if (subproblem is unfeasible) **then**
    return
  end if
  if (subproblem is unbounded) **then**
    get a direction of unboundedness $\mathbf{v}_0$; // or an unbounded ray;
    add to the master problem the cut $(\mathbf{b}-\mathbf{B}\mathbf{y})^T\mathbf{v}_0\leqslant 0$;
  else
    let $\mathbf{v}_0$ be a basic optimal solution;
    add to the master problem the cut $z\geqslant\mathbf{q}^T\mathbf{y}+(\mathbf{b}-\mathbf{B}\mathbf{y})^T\mathbf{v}_0$;
    $u\leftarrow\min\{u,\mathbf{q}^T\mathbf{y}_0+(\mathbf{b}-\mathbf{B}\mathbf{y}_0)^T\mathbf{v}_0\}$;
  end if
  solve the master problem;
  $l\leftarrow z$;
end while
return $\mathbf{v}_0$.

References

[1] Desaulniers, G., J. Desrosiers, Solomon, M. M., *Column Generation*, Springer, US, 2005.

[2] G. Carpaneto, M. Dell'Amico, M. Fischetti, P. Toth. *A Branch and Bound Algorithm for the Multiple Depot Vehicle Scheduling Problem,* Networks 19, 531–548, 1989. `doi:10.1002/net.3230190505`.

[3] Held, S., W. Cook, E. C.. Sewell, *Maximum-weight stable sets and safe lower bounds for graph coloring*, Math. Prog. Comp, 4, 363–381, 2012.

[4] M. A. Forbes, J. N. Holt, A. M. Watts. *An exact algorithm for multiple depot bus scheduling,* European Journal of Operational Research 72, 115–124, 1994.

[5] Lubbecke, M. and Desrosiers, J., *Selected topics in column generation*, Operations Research, 53(6), 1007–1023, 2005.

[6] Mehrotra, A. and Trick, M., *A column generation approach for graph coloring*, INFORMS Journal On Computing, 8(4), 344–354, 1996.

[7] A.-S. Pepin, G. Desaulniers, A. Hertz, D. Huisman. *A comparison of five heuristics for the multiple depot vehicle scheduling problem,* Journal of Scheduling 12, 17–30, 2009. `doi: 10.1007/s10951-008-0072-x`.

[8] C.-P. Teo, J. Sethuraman. *On a cutting plane heuristic for the stable roommates problem and its applications,* European Journal of Operational Research 123, 195–205, 2000.