Seminar 7

1. Branch-and-Bound Algorithm - an Example.

Consider the following ILP problem

$$(P)\begin{cases} \max & 5x_1+6x_2 \\ \text{s. t.} & 10x_1+3x_2 \leqslant 52 \\ & 2x_1+3x_2 \leqslant 18 \\ & x_1,x_2 \in \mathbb{Z}_+ \end{cases}.$$

Its relaxation is

$$(R)\begin{cases} \max & 5x_1+6x_2 \\ \text{s. t.} & 10x_1+3x_2 \leqslant 52 \\ & 2x_1+3x_2 \leqslant 18 \\ & x_1,x_2 \geqslant 0 \end{cases}.$$

We solve (R) and find the optimal solution $x_1 = 17/4, x_2 = 19/6$ with $z_0 = 161/4$. From this solution which is not feasible for (P) we can branch on x_1 and get two new (ILP) problems (P_1) and (P_2), those LP relaxations are

$$(R_1)\begin{cases} \max & 5x_1+6x_2 \\ \text{s. t.} & 10x_1+3x_2 \leqslant 52 \\ & 2x_1+3x_2 \leqslant 18 \\ & x_1 \geqslant 5 \\ & x_1,x_2 \geqslant 0 \end{cases} \quad (R_2)\begin{cases} \max & 5x_1+6x_2 \\ \text{s. t.} & 10x_1+3x_2 \leqslant 52 \\ & 2x_1+3x_2 \leqslant 18 \\ & x_1 \leqslant 4 \\ & x_1,x_2 \geqslant 0 \end{cases}$$

By solving (R_1) we get $x_1 = 5, x_2 = 2/3$ with $z_0 = 29$. We must branch on x_2, and we obtain two new ILP problems those relaxations are

$$(R_3)\begin{cases} \max & 5x_1+6x_2 \\ \text{s. t.} & 10x_1+3x_2 \leqslant 52 \\ & 2x_1+3x_2 \leqslant 18 \\ & x_1 \geqslant 5 \\ & x_2 \geqslant 1 \\ & x_1,x_2 \geqslant 0 \end{cases} \quad (R_4)\begin{cases} \max & 5x_1+6x_2 \\ \text{s. t.} & 10x_1+3x_2 \leqslant 52 \\ & 2x_1+3x_2 \leqslant 18 \\ & x_1 \geqslant 5 \\ & x_2 \leqslant 0 \\ & x_1,x_2 \geqslant 0 \end{cases}$$

Problem (R_3) is infeasible. By solving (R_4), we get $x_1 = 26/5, x_2 = 0$ with $z_0 = 26$. After branching on x_2 we have two more problems

$$(R_5)\begin{cases} \max & 5x_1+6x_2 \\ \text{s. t.} & 10x_1+3x_2 \leqslant 52 \\ & 2x_1+3x_2 \leqslant 18 \\ & x_1 \geqslant 5 \\ & x_2 \leqslant 0 \\ & x_1 \geqslant 6 \\ & x_1,x_2 \geqslant 0 \end{cases} \quad (R_6)\begin{cases} \max & 5x_1+6x_2 \\ \text{s. t.} & 10x_1+3x_2 \leqslant 52 \\ & 2x_1+3x_2 \leqslant 18 \\ & x_1 \geqslant 5 \\ & x_2 \leqslant 0 \\ & x_1 \leqslant 5 \\ & x_1,x_2 \geqslant 0 \end{cases}$$

Problem (R_5) is infeasible. By solving (R_6), we get $x_1 = 5, x_2 = 0$ with $z_0 = 25$. At this point we first update the best so-far solution: $\overline{\mathbf{x}} = (5,0)^T$, $\overline{z} = 25$.

It's time to move on the other branch of our tree an solve problem (R_2): we obtain $x_1 = 4, x_2 = 10/3$, $z_0 = 40$, $z_0 > \overline{z}$ means it is worth trying this new sub-tree; branching on x_2 we get two new problems whose relaxations are

$$(R_7)\begin{cases} \max & 5x_1 + 6x_2 \\ \text{s. t.} & 10x_1 + 3x_2 \leqslant 52 \\ & 2x_1 + 3x_2 \leqslant 18 \\ & x_1 \leqslant 4 \\ & x_2 \leqslant 3 \\ & x_1, x_2 \geqslant 0 \end{cases} \qquad (R_8)\begin{cases} \max & 5x_1 + 6x_2 \\ \text{s. t.} & 10x_1 + 3x_2 \leqslant 52 \\ & 2x_1 + 3x_2 \leqslant 18 \\ & x_1 \leqslant 4 \\ & x_2 \geqslant 4 \\ & x_1, x_2 \geqslant 0 \end{cases}$$

Problem (R_7) has as an optimal solution $x_1 = 4, x_2 = 3$ with $z_0 = 38$. We can update the so-far optimal solution: $\overline{\mathbf{x}} = (4,3)^T$, $\overline{z} = 38$.

By solving problem (R_8) we get $x_1 = 3, x_2 = 4$, and $z_0 = 39$, which is also an integer solution which becomes the best so far: $\overline{\mathbf{x}} = (3,4)^T$, $\overline{z} = 39$.

We fathomed all nodes of the tree: the optimal solution is $\overline{\mathbf{x}} = (3,4)^T$, $\overline{z} = 39$.

2. Cutting Plane Algorithm (with Gomory fractional cuts) - an Example.

Consider the following ILP problem

$$(P)\begin{cases} \min & x_1 - 2x_2 \\ \text{s. t.} & -4x_1 + 6x_2 \leqslant 9 \\ & x_1 + x_2 \leqslant 4 \\ & x_1, x_2 \in \mathbb{Z}_+ \end{cases}.$$

The natural LP relaxation is

$$(R)\begin{cases} \min & x_1 - 2x_2 \\ \text{s. t.} & -4x_1 + 6x_2 \leqslant 9 \\ & x_1 + x_2 \leqslant 4 \\ & x_1, x_2 \geqslant 0 \end{cases}.$$

Its standard form is

$$(R_1)\begin{cases} \min & x_1 - 2x_2 \\ \text{s. t.} & -4x_1 + 6x_2 + x_3 = 9 \\ & x_1 + x_2 + x_4 = 4 \\ & x_1, x_2, x_3, x_4 \geqslant 0 \end{cases}.$$

We solve (R_1) using Simplex algorithm (Two Phase Method is not really necessary here). The last tableau is

Table 1: Last Tableau for (R_1).

	x_1	x_2	x_3	x_4	RHS
x_2	0	1	1/10	4/10	25/10
x_1	1	0	-1/10	6/10	15/10
z	0	0	3/10	2/10	35/10

Let us consider the first equation (corresponding to a non-integer value):

$$x_2 + \frac{1}{10}x_3 + \frac{4}{10}x_4 = \frac{25}{10}.$$

The associated Gomory fractional cut is $x_2 \leqslant 2$. We add the constraint to our LP relaxation and get

$$(R_2)\begin{cases} \min & x_1 - 2x_2 \\ \text{s. t.} & -4x_1 + 6x_2 + x_3 = 9 \\ & x_1 + x_2 + x_4 = 4 \\ & x_2 + x_5 = 2 \\ & x_1, x_2, x_3, x_4, x_5 \geqslant 0 \end{cases}.$$

We solve (R_2):

Table 2: Last Tableau for (R_2).

	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	0	0	1	2
x_4	0	0	1/4	1	-5/2	5/4
x_1	1	0	-1/4	0	3/2	3/4
z	0	0	1/4	0	1/2	13/4

Let us consider the last equation (corresponding to a non-integer value):

$$x_1 - \frac{1}{4}x_3 + \frac{3}{2}x_5 = \frac{3}{4}.$$

The new Gomory cut is $x_1 - x_3 + x_5 \leqslant 0$, which, in terms of original variables becomes

$$-3x_1 + 5x_2 \leqslant 7.$$

We add the constraint to our LP relaxation (R_2) and get

$$(R_3)\begin{cases} \min & x_1 - 2x_2 \\ \text{s. t.} & -4x_1 + 6x_2 + x_3 = 9 \\ & x_1 + x_2 + x_4 = 4 \\ & x_2 + x_5 = 2 \\ & -3x_1 + 5x_2 + x_6 = 7 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geqslant 0 \end{cases}.$$

Solving (R_3) and find a feasible integer solution $x_1 = 1, x_2 = 2$, which must be an optimal feasible (integer) solution for our original problem.

Table 3: Last Tableau for (R_3).

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	0	0	1	0	2/3	-4/3	1
x_4	0	0	0	1	-8/3	1/3	1
x_1	1	0	0	0	5/3	-1/3	1
x_2	0	1	0	0	1	0	2
z	0	0	0	0	1/3	1/3	3