

Seminar 5

1. Write down the corresponding dual problems for the following LP problems and verify that the dual of the dual is the primal.

$$(a) \begin{cases} \min & x_1 - x_2 \\ \text{s. t.} & 2x_1 + 3x_2 - x_3 + x_4 \leq 0 \\ & 3x_1 + x_2 + 4x_3 - 2x_4 \geq 3 \\ & -x_1 - x_2 + 2x_3 + x_4 = 6 \\ & x_1 \leq 0, x_2, x_3 \geq 0. \end{cases} \quad (b) \begin{cases} \max & x_1 - 2x_2 + 2x_3 \\ \text{s. t.} & x_1 + 2x_2 - x_3 + x_4 \geq 0 \\ & 4x_1 + 3x_2 + 4x_3 - 2x_4 \leq 3 \\ & x_1 - x_2 + x_3 + x_4 = 1 \\ & x_2, x_3 \geq 0. \end{cases}$$

$$(c) \begin{cases} \min & 2x_1 + x_2 - x_3 \\ \text{s. t.} & x_1 + 2x_2 + x_3 + x_4 \geq 4 \\ & 2x_1 - x_2 + 3x_3 - x_4 \geq 2 \\ & x_1 - x_2 + x_3 + 3x_4 \geq 5 \\ & x_1, x_3 \geq 0, x_4 \leq 0. \end{cases}$$

2. Find the duals of the following LP problems

$$(a) \begin{cases} \text{minimize} & z = \mathbf{c}^T \mathbf{x} \\ \text{subject to} & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{u}^1 \leq \mathbf{x} \leq \mathbf{u}^2. \end{cases} \quad (b) \begin{cases} \text{minimize} & z = \mathbf{c}^T \mathbf{x} \\ \text{subject to} & \mathbf{b}^1 \leq \mathbf{Ax} \leq \mathbf{b}^2 \\ & \mathbf{x} \geq \mathbf{0}. \end{cases}$$

3. Consider the following process. Starting with an LP problem

$$\begin{aligned} & \text{minimize} && z = \mathbf{c}^T \mathbf{x} \\ & \text{subject to} && \mathbf{Ax} \geq \mathbf{b}, \\ & && \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

first form its dual

$$\begin{aligned} & \text{maximize} && z = \mathbf{b}^T \mathbf{y} \\ & \text{subject to} && \mathbf{A}^T \mathbf{y} \leq \mathbf{c}, \\ & && \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

then replace the maximization in the dual with a minimization and get a new LP problem, which can be written like follows:

$$\begin{aligned} & \text{minimize} && z = \mathbf{b}^T \mathbf{y} \\ & \text{subject to} && -\mathbf{A}^T \mathbf{y} \geq -\mathbf{c} \\ & && \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

Obviously, since an LP problem can be identified with its data $(\mathbf{A}, \mathbf{b}, \mathbf{c})$, the above process is a transformation T on the space of data:

$$(\mathbf{A}, \mathbf{b}, \mathbf{c}) \xrightarrow{T} (-\mathbf{A}^T, -\mathbf{c}, \mathbf{b}).$$

Let $z_*(\mathbf{A}, \mathbf{b}, \mathbf{c})$ denote the optimal objective function value of the LP problem having data $(\mathbf{A}, \mathbf{b}, \mathbf{c})$. By strong dualization theorem, together with the fact that maximization dominates minimization, we must have

$$z_*(\mathbf{A}, \mathbf{b}, \mathbf{c}) \geq z_*(-\mathbf{A}^T, -\mathbf{c}, \mathbf{b}).$$

Now if we repeat this process we have

$$(\mathbf{A}, \mathbf{b}, \mathbf{c}) \xrightarrow{T} (-\mathbf{A}^T, -\mathbf{c}, \mathbf{b}) \xrightarrow{T} (\mathbf{A}, -\mathbf{b}, -\mathbf{c}) \xrightarrow{T} (-\mathbf{A}^T, \mathbf{c}, -\mathbf{b}) \xrightarrow{T} (\mathbf{A}, \mathbf{b}, \mathbf{c}).$$

Therefore,

$$z_*(\mathbf{A}, \mathbf{b}, \mathbf{c}) \geq z_*(-\mathbf{A}^T, -\mathbf{c}, \mathbf{b}) \geq z_*(\mathbf{A}, -\mathbf{b}, -\mathbf{c}) \geq z_*(-\mathbf{A}^T, \mathbf{c}, -\mathbf{b}) \geq z_*(\mathbf{A}, \mathbf{b}, \mathbf{c}).$$

Since the first and the last expressions in this chain of inequalities are equal, all these inequalities must be equalities.

- (i) This outcome, while could happen sometimes, it isn't always true. What is the error in this logic?
- (ii) Can you state a correct theorem that follows this line of reasoning?

4. Consider the following LP problem

$$\begin{aligned} & \text{minimize} && z = \mathbf{c}^T \mathbf{x} \\ & \text{subject to} && \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & && \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

Assume that this problem and its dual are feasible. Let \mathbf{x}_* be an optimal solution to the primal and let \mathbf{y}_* be an optimal solution to the dual. For each of the following changes, describe what effect they have on \mathbf{x}_* and \mathbf{y}_* if any.

- (i) The vector \mathbf{c} is multiplied by $\lambda > 0$.
- (ii) The k th equation is multiplied by $\lambda \in \mathbb{R}$.
- (iii) The right-hand side \mathbf{b} is multiplied by $\lambda \in \mathbb{R}$.

5. Consider the following two LP problems

$$(P_1) \begin{cases} \text{minimize} & z = \tilde{\mathbf{c}}^T \mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{cases} \quad (P_2) \begin{cases} \text{minimize} & z = \bar{\mathbf{c}}^T \mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{cases}$$

Let $\tilde{\mathbf{x}}$ be an optimal solution to (P_1) and $\bar{\mathbf{x}}$ be an optimal solution to (P_2) , assumed to exist. Prove that $(\tilde{\mathbf{c}} - \bar{\mathbf{c}})^T(\tilde{\mathbf{x}} - \bar{\mathbf{x}}) \leq 0$.

6. Using duality theory find the optimal solution to the following linear program:

$$\begin{aligned} & \min && z = x_1 + 2x_2 + \dots + nx_n \\ & \text{s. t.} && x_1 \geq 1 \\ & && x_1 + 2x_2 \geq 2 \\ & && \dots \\ & && x_1 + 2x_2 + \dots + nx_n \geq n \\ & && x_1, x_2, \dots, x_n \geq 0 \end{aligned}$$

7. Consider the following LP problem

$$(P) \begin{cases} \max & z = x_1 + 3x_2 + x_3 + x_4 \\ \text{s. t.} & 2x_1 + x_2 - x_3 - x_4 \geq 3 \\ & x_1 + x_2 + x_3 + x_4 \leq 4 \\ & -x_2 + x_4 \geq 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

The solution $x_1 = 2, x_2 = 0.5, x_3 = 0, x_4 = 1.5$ has been proposed as an optimal solution of the above problem. Try to verify this assertion by means of the duality theory (build the dual problem and use complementary slackness/weak duality/strong duality theorem).