

## Seminar 4

### 1. Two phase method

Consider the following LP problem

$$(1) \quad \begin{cases} \max & 2x_1 - x_2 + 2x_3 \\ \text{s. t.} & x_1 + x_2 + x_3 \leq 6 \\ & -x_1 + x_2 \leq -1 \\ & -x_1 + x_3 \leq -1 \\ & x_1, x_2, x_3 \geq 0 \end{cases}$$

We add the slack variables  $x_4, x_5, x_6$ :

$$\begin{cases} \max & 2x_1 - x_2 + 2x_3 \\ \text{s. t.} & x_1 + x_2 + x_3 + x_4 = 6 \\ & -x_1 + x_2 + x_5 = -1 \\ & -x_1 + x_3 + x_6 = -1 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{cases}$$

We convert the problem to the standard form:

$$\begin{cases} \min & z = -2x_1 + x_2 - 2x_3 \\ \text{s. t.} & x_1 + x_2 + x_3 + x_4 = 6 \\ & x_1 - x_2 - x_5 = 1 \\ & x_1 - x_3 - x_6 = 1 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{cases}$$

Obviously,  $\{x_4, x_5, x_6\}$  is not an initial feasible solution.

#### Phase I

We have to introduce two artificial variables  $y_1$  and  $y_2$  for the last two constraints. Together with  $x_4$ , they will form a initial basic feasible solution. However, we must also change the objective function, therefore we get a new problem:

$$(2) \quad \begin{cases} \min & z' = y_1 + y_2 \\ \text{s. t.} & x_1 + x_2 + x_3 + x_4 = 6 \\ & x_1 - x_2 - x_5 + y_1 = 1 \\ & x_1 - x_3 - x_6 + y_2 = 1 \\ & x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2 \geq 0 \end{cases}$$

In this problem we must write  $y_1$  and  $y_2$  in terms of the nonbasic variables:

$$y_1 = 1 - x_1 + x_2 + x_5, \quad y_2 = 1 - x_1 + x_3 + x_6.$$

The objective function becomes

$$z' = y_1 + y_2 = -2x_1 + x_2 + x_3 + x_5 + x_6 + 2.$$

The problem we have to solve in Phase I will be

$$\begin{cases} \min & z' = -2x_1 + x_2 + x_3 + x_5 + x_6 + 2 \\ \text{s. t.} & x_1 + x_2 + x_3 + x_4 = 6 \\ & x_1 - x_2 - x_5 + y_1 = 1 \\ & x_1 - x_3 - x_6 + y_2 = 1 \\ & x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2 \geq 0 \end{cases}$$

From now on we can use the simplex algorithm:

Table 1: Phase I - First Simplex Tableau.

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$y_1$	$y_2$	RHS	
$x_4$	1	1	1	1	0	0	0	0	6	6/1
$y_1$	1	-1	0	0	-1	0	1	0	1	$\frac{1}{1}$ ← min
$y_2$	1	0	-1	0	0	-1	0	1	1	1/1
$z'$	-2	1	1	0	1	1	0	0	-2	

Table 2: Phase I - Second Simplex Tableau.

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$y_1$	$y_2$	RHS	
$x_4$	0	2	1	1	1	0	-1	0	5	5/2
$x_1$	1	-1	0	0	-1	0	1	0	1	
$y_2$	0	1	-1	0	1	-1	-1	1	0	$\frac{0}{1}$ ← min
$z'$	0	-1	1	0	-1	1	2	0	0	

Table 3: Phase I - Third Simplex Tableau.

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$y_1$	$y_2$	RHS
$x_4$	0	0	3	1	-1	2	1	-2	5
$x_1$	1	0	-1	0	0	-1	0	1	1
$x_2$	0	1	-1	0	1	-1	0	1	0
$z'$	0	0	0	0	0	0	1	1	0

The third tableau is optimal, the optimal objective value is zero, and the artificial variables are non-basic (so they are all zero). This says that the original problem is feasible.

## Phase II

We remove the columns corresponding to the artificial variables and restate the original objective function.

Table 4: Restated Simplex Tableau.

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	RHS
$x_4$	0	0	3	1	-1	2	5
$x_1$	1	0	-1	0	0	-1	1
$x_2$	0	1	-1	0	1	-1	0
$z$	-2	1	-2	0	0	0	0

Since the reduced cost of basic variables are not all zeros, this tableau is not in the proper Simplex form. We have to replace these variables in the objective function; this means to represent the linear program in terms of the current basis:

$$x_1 = x_3 + x_6 + 1, \quad x_2 = x_3 - x_5 + x_6,$$

$$z = -2x_1 + x_2 - 2x_3 = -3x_3 - x_5 - x_6 - 2.$$

Table 5: Phase II - First Simplex Tableau.

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	RHS	
$x_4$	0	0	<u>3</u>	1	-1	2	5	$\frac{5}{3} \leftarrow \min$
$x_1$	1	0	-1	0	0	-1	1	
$x_2$	0	1	-1	0	1	-1	0	
$z$	0	0	<u>-3</u>	0	-1	-1	2	

Table 6: Phase II - Second Simplex Tableau.

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	RHS	
$x_3$	0	0	1	1/3	-1/3	2/3	5/3	
$x_1$	1	0	0	1/3	-1/3	-1/3	8/3	
$x_2$	0	1	0	1/3	<u>2/3</u>	-1/3	5/3	$\frac{5}{2} \leftarrow \min$
$z$	0	0	0	1	<u>-2</u>	1	7	

Table 7: Phase II - Second Simplex Tableau.

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	RHS
$x_3$	0	1/2	1	1/2	0	1/2	5/2
$x_1$	1	1/2	0	1/2	0	-1/2	7/2
$x_5$	0	3/2	0	1/2	1	-1/2	5/2
$z$	0	3	0	2	0	0	12

The current basic feasible solution is an optimal one:  $x_1 = 7/2, x_2 = 0, x_3 = 5/2, x_4 = 0, x_5 = 5/2, x_6 = 0$ ; the optimal value of the objective function is  $-12$  (this is the value of the standard form problem).

Obviously, the objective function value of the problem (2) is non-negative.

If, during the run of the Simplex algorithm in Phase I, we come out with a tableau whose value of the objective function equals zero, then this will be also the final value of the objective function. This situation can occur when the tableau is optimal or not (see the second Simplex tableau from the Phase I in the above example).

At such a moment we can stop the running of the Simplex algorithm. If all the artificial variables are nonbasic we can go to the Phase II as in the above example. Otherwise suppose that the  $i$ th row is labeled by an artificial variable  $x_k$ :

- (i) The variable  $x_k$  can be replaced in the basis with any non-basic variable  $x_j$  for which  $\hat{a}_{ij} \neq 0$ .
- (ii) If all the candidates,  $x_j$ , for replacing  $x_k$  are artificial variables, then the constraint corresponding to the  $i$ th row is redundant (why?) and can be removed from the tableau.