Seminar 2

Standard Forms and Convexity

1. (The full row rank assumption on matrix A**)** Let $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n_+ \ : \ \mathbf{Ax} = \mathbf{b}\} \neq \varnothing$, where $\mathbf{A}$ is a matrix of dimensions $m \times n$, with rows $\mathbf{a}'_1, \dots \mathbf{a}'_m$. Suppose that $rank(\mathbf{A}) = k < m$ and that the rows $\mathbf{a}'_{i_1}, \dots, \mathbf{a}'_{i_k}$ are linearly independent. Consider the polyhedron $\mathcal{Q} = \{\mathbf{x} \in \mathbb{R}^n_+ \ : \ \mathbf{a}'_{i1}\mathbf{x} = b_{i_1}, \dots, \mathbf{a}'_{ik}\mathbf{x} = b_{i_k}\}$. Prove that $\mathcal{P} = \mathcal{Q}$.

(As a consequence: if the feasible set is not empty, an LP problem in standard form can be reduced to an equivalent standard form problem with a matrix having full row rank.)

2. Convert the following LP problems to canonical form

$$\left\{\begin{array}{lrl} \max & x_1 + x_2 - 2x_4 & \\ \text{s. t.} & 3x_1 + x_2 - x_4 & = 2 \\ & -x_1 + 2x_2 + x_3 - x_4 & \geqslant 1 \\ & 2x_1 + 2x_2 - x_3 + 2x_4 & \leqslant 5 \\ & x_1 \geqslant 0, x_2 \leqslant 3, x_3 \geqslant 1, x_4 \in \mathbb{R} & \end{array}\right. \qquad \left\{\begin{array}{lrl} \min & 3x_1 - x_2 + 2x_3 & \\ \text{s. t.} & 2x_1 + x_2 - x_3 & \geqslant 4 \\ & x_1 - 2x_2 + 4x_3 & = 3 \\ & 3x_1 + 3x_2 + 2x_3 & \leqslant 2 \\ & x_1 \leqslant 1, x_2 \in \mathbb{R}, x_3 \leqslant 3 & \end{array}\right.$$

3. Convert the following LP problems to standard form

$$\left\{\begin{array}{lrl} \max & 3x_1 + 5x_2 - 4x_3 & \\ \text{s. t.} & 7x_1 - 2x_2 - 3x_3 & \geqslant 4 \\ & -2x_1 + 4x_2 + 8x_3 & = -3 \\ & 5x_1 - 3x_2 - 2x_3 & \leqslant 9 \\ & x_1 \geqslant 1, x_2 \leqslant 7, x_3 \geqslant 0 & \end{array}\right. \qquad \left\{\begin{array}{lrl} \min & x_1 - 5x_2 - 7x_3 & \\ \text{s. t.} & 3x_1 + 4x_2 - 9x_3 & = 3 \\ & 5x_1 - 2x_2 + 6x_3 & \geqslant 5 \\ & 7x_1 + 3x_2 + 5x_3 & \leqslant 6 \\ & x_1 \geqslant -1, x_2, x_3 \in \mathbb{R} & \end{array}\right.$$

4. Prove Theorem 2.1 (iii) and/or (iv) from Lecture 2.

5. Consider the LP problem $\max\{\mathbf{c}^T\mathbf{x} \ : \ \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geqslant \mathbf{0}\}$ (matrix A has full row rank). True or false:

(i) the set of optimal solutions is bounded?

(ii) if the problem has several optimal solutions, there exist at least two basic feasible solutions that are optimal?

(iii) if the problem is bounded the feasible region is a bounded set?

(iiv) if the problem is ubounded the feasible region is an unbounded set?

(*Hint:* Recall the definitions of bounded/unbounded LP problem.)

6*. Prove that the set $\{x \ : \ \mathbf{Ax} < \mathbf{b}\}$ doesn't contain any extreme points.

Basic feasible solutions.

7. For the following polyhedra, write the systems in standard form, and determine all the basic solutions (feasible and infeasible).

$$\textbf{(a)} \left\{\begin{array}{rl} 2x_1 + x_2 & \leqslant 80 \\ x_1 + x_2 & \leqslant 60 \\ x_1 & \leqslant 40 \\ x_1, x_2 \geqslant 0 & \end{array}\right. \qquad \textbf{(b)} \left\{\begin{array}{rl} 2x_1 + x_2 & \leqslant 4 \\ x_1 + x_2 & \leqslant 3 \\ x_1, x_2 \geqslant 0 & \end{array}\right.$$

8. For the following LP problems, write the systems in standard form, determine all the basic solutions (feasible and infeasible), and find optimal basic solutions (if any).

$$\textbf{(a)}\left\{\begin{array}{lrl} \min & 3x_1 - x_2 & \\ \text{s. t.} & 2x_1 + 2x_2 & \leqslant 6 \\ & x_1 - x_2 & \leqslant 3 \\ & x_1, x_2 \geqslant 0 & \end{array}\right. \qquad \textbf{(b)}\left\{\begin{array}{lrl} \min & 3x_1 + 2x_2 & \\ \text{s. t.} & x_1 - x_2 & \leqslant 3 \\ & -x_1 + 2x_2 & \leqslant 4 \\ & x_1, x_2 \geqslant 0 & \end{array}\right.$$