Seminar 12

1. Show that the following matrix $\mathbf{A}$ is not totally unimodular, but the polyhedron $\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{A}\mathbf{x} = \mathbf{b}\}$ is integral for every $\mathbf{b} \in \mathbb{Z}^3$.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Let $\mathbf{A}$ be an $m \times n$ matrix. Prove that the following are equivalent:

(i) $\mathbf{A}$ is totally unimodular.

(ii) $\mathbf{A}^T$ is totally unimodular.

(iii) The matrix $[I_m\ \mathbf{A}]$ is unimodular.

(iv) The matrix $[\mathbf{A}\ -\mathbf{A}]$ is totally unimodular.

(v) Any matrix $\mathbf{A}'$ obtained from $\mathbf{A}$ by changing the signs of all entry in some rows is totally unimodular.

3. Prove that the extreme points of $\{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{A}\mathbf{x} \leqslant \mathbf{b}\}$ are integral iff the extreme points of $\{\mathbf{y} \in \mathbb{R}^{m+n}_+ : [I_m\ \mathbf{A}]\mathbf{y} = \mathbf{b}\}$ are integral.

4. $\mathbf{A} \in \mathbb{R}^{m \times n}$ is totally unimodular iff the following matrix is also totally unimodular

$$\widetilde{\mathbf{A}} = \begin{bmatrix} I_n \\ -I_n \\ \mathbf{A} \\ -\mathbf{A} \end{bmatrix}.$$

5. Let $\mathbf{A}$ be an integral $m \times n$ matrix. Then $\mathbf{A}$ is totally unimodular iff each (and every) of the following polyhedra is integral $\{\mathbf{x} : \mathbf{x} \leqslant \mathbf{d}, \mathbf{A}\mathbf{x} \leqslant \mathbf{b}\}, \{\mathbf{x} : \mathbf{x} \leqslant \mathbf{d}, \mathbf{A}\mathbf{x} \geqslant \mathbf{b}\}, \{\mathbf{x} : \mathbf{x} \geqslant \mathbf{d}, \mathbf{A}\mathbf{x} \leqslant \mathbf{b}\}, \{\mathbf{x} : \mathbf{x} \geqslant \mathbf{d}, \mathbf{A}\mathbf{x} \geqslant \mathbf{b}\}$, for every given $\mathbf{b} \in \mathbb{Z}^m$, and $\mathbf{d} \in \mathbb{Z}^n$.

6. Let $\mathbf{A}$ be an integral $m \times n$ matrix. Then A is totally unimodular iff for all integral vectors $\mathbf{b}$ and $\mathbf{c}$. the duality equation

$$\max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \geqslant \mathbf{0}, \mathbf{A}\mathbf{x} \leqslant \mathbf{b}\} = \min\{\mathbf{b}^T\mathbf{y} : \mathbf{y} \geqslant \mathbf{0}, \mathbf{A}^T\mathbf{y} \geqslant \mathbf{c}\}$$

is achieved by integral vectors $\mathbf{x}$ and $\mathbf{y}$ (provided the problems are bounded).

7. Consider the following two matrices

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

and $\mathcal{P}_i = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{A}_i\mathbf{x} \leqslant \mathbf{0}\}$.

(a) Prove that $\mathcal{P}_1 = \mathcal{P}_2$.

(b) Show that the system $\mathbf{A}_1\mathbf{x} \leqslant \mathbf{0}$ is TDI but $\mathbf{A}_2\mathbf{x} \leqslant \mathbf{0}$ is not.

8. Consider the following two polyhedra:

$$\mathcal{P}_1 = \{(x_1, x_2) \in \mathbb{R}^2_+ \ : \ x_1 + 2x_2 \leqslant 6, 2x_1 + x_2 \leqslant 6\}$$

$$\mathcal{P}_2 = \{(x_1, x_2) \in \mathbb{R}^2_+ \ : \ x_1 + 2x_2 \leqslant 6, 2x_1 + x_2 \leqslant 6, x_1 + x_2 \leqslant 4, x_1 \leqslant 3, x_2 \leqslant 3\}$$

Prove that

(a) $\mathcal{P}_1 = \mathcal{P}_2$.

(b) $\mathcal{P}_1$ is not TDI.

(c) $\mathcal{P}_2$ is TDI.