Seminar 11

Silicon I'ntl ACME has developed a new type of processor that will enable it to begin producing and marketing a PC if it so desires. Alternatively, it can sell the rights to the processor for \$15 million. If the company chooses to build PCs, it has sufficient access to retail market that it can guarantee sales of 10,000 computers. On the other hand, if this computer catches on, the company can sell 100,000 machines. The cost of setting up the assembly line is \$6 million. The net profit for one PC is \$600.

The company can pay \$1 million for a market survey which will reveal the chances of a new PC on the market. From past experience we know that if a new product will have success on the market, then the survey will find out this with probability 2/3, and if a new product will be a failure, then the survey will find out this with probability 3/4.

1. Decision Analysis without Experimentation.

- the decision maker has a number of alternatives: A_i, $i = \overline{1, m}$;
- the states of nature with estimated probabilities are $P(S_j) = q_j > 0$, $\sum_{j=1}^{n} q_j = 1$, where q_j are the "a priori" probabilities;
- each pair (alternative, state) has a payoff: p_{ij};
- the table of payoffs is

Alternatives:	States of nature:			
	S_1	S_2	$\dots$	S_n
A_1	p_{11}	p_{12}	$\dots$	p_{1n}
A_2	p_{21}	p_{22}	$\dots$	p_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	p_{m1}	p_{m2}	$\dots$	p_{mn}
probabilities:	q_1	q_2	$\dots$	q_n

Alternatives:	States of nature:	
	retail guarantee only (S_1)	market succes (S_2)
Produce PC (A_1)	0\$	54 mil.\$
Sell the rights (A_2)	15 mil.\$	15 mil.\$
Probabilities:	0.5	0.5

Used criteria

- **(Maximin)** $\max_i (\min_j p_{ij})$.

  For each $1 \leqslant i \leqslant m$, fie $p_{ij_i} = \min_{1 \leqslant j \leqslant n} p_{ij}$, we choose A_{i_0}, where $p_{i_0 j_{i_0}} = \max_{1 \leqslant i \leqslant m} p_{ij_i}$.

  In our case we choose A_2.

- **(Maximum likelihood)** we choose first the maximum likelihood state; for this state we choose the alternative which gives the maximum payoff:

  $$\text{let } S_{j_0} \text{ with } q_{j_0} = \max_{1 \leqslant j \leqslant n} q_j, \text{ choose } A_{i_0}, \text{ where } p_{i_0 j_0} = \max_{1 \leqslant i \leqslant m} p_{ij_0}.$$

  In our case we can equally choose A_1 or A_2.

- **(Bayes Rule)** We compute the expected payoff for each alternative; then we choose the maximum expected payoff alternative

$$\text{choose } A_{i_0}, \text{ with } \mathbb{E}[A_{i_0}] = \max_{1 \leqslant i \leqslant m} \mathbb{E}[A_i],$$

$$\text{where } \mathbb{E}[A_i] = \sum_{j=1}^{n} q_j p_{ij} \text{ is the expected payoff of } A_i.$$

  For our example $\mathbb{E}[A_1] = 0 \cdot 0.5 + 54 \cdot 0.5 = \$27$ mil., $\mathbb{E}[A_2] = 15 \cdot 0.5 + 15 \cdot 0.5 = \$15$ mil.; we have to choose A_1.

Senzitivity Analysis with Bayes Rule.

Choose two states (S_k, S_l), and parameters for their probabilities [1]:

$$P(S_k) = q, P(S_l) = q' = 1 - q - \sum_{j \neq k,l} q_j,$$

where $0 \leqslant q \leqslant 1 - \sum_{j \neq k,l} q_j$. The *critical point* for the pair $(A_i, A_{i'})$ is the solution of the equation:

$$\mathbb{E}[A_i] = qp_{ik} + q'p_{il} + \sum_{j \neq k,l}^{n} q_j p_{ij} = \mathbb{E}[A_{i'}] = qp_{i'k} + q'p_{i'l} + \sum_{j \neq k,l}^{n} q_j p_{i'j}$$

For our example let $q = P(S_1)$, $\mathbb{E}[A_1] = 0 \cdot q + 54 \cdot (1-q) = 54 - 54q$, and $\mathbb{E}[A_2] = 15 \cdot q + 15 \cdot (1-q) = 15$. The equation is $54 - 54q = 15 \Rightarrow q = \frac{39}{54} \cong 0.722$.

If $q = P(S_1) > 0.722$ it is better to sell the processor otherwise ($q < 0.722$) it is better to produce and market the PC.

2. Decision Analysis with Experimentation.

An experiment can reveal the true state of nature for the system or, at least, improve the a priori probability estimates. Let F the possible value of the experiment: a random variable with values: $\{f_1, f_2, \ldots, f_t\}$. (Also called *Finding* in course 10). Let S the random variable which gives the state of nature (Former *State* variable.)

If we know all the conditioning probabilities $P(F = f_k | S = S_j)$, then we can compute the a posteriori probabilities:

$$P(S = S_j | F = f_k) = \frac{P(S = S_j \wedge F = x_k)}{P(F = f_k)} = \frac{P(F = f_k | S = S_j) \cdot q_j}{\sum_{l=1}^{n} P(F = f_k | S = S_l) \cdot q_l}$$

For each possible outcome of the experiment f_k, using Bayes' rule we can decide which alternative to choose:

$$1 \leqslant i_0 \leqslant m \text{ s.t. } (\mathbb{E}[A_{i_0} | F = f_k]) = \max_{1 \leqslant i \leqslant m} (\mathbb{E}[A_i | F = f_k]), \text{ where}$$

$$\mathbb{E}[A_i | F = f_k] = \sum_{j=1}^{n} P(S = S_j | F = f_k) \cdot p_{ij}.$$

Estimating the value of the experiment.

[1] The other probabilities, except $P(S_k)$ and $P(S_l)$, remain the same.

- (*EVPI - Expected Value of Perfect Information*)

  Perfect information means to know the true state of nature. The expected value of perfect information is

  $$EVPI = EPPI - EPWE,$$

  where the expected payoff with perfect information is $EPPI = \sum_{j=1}^{n} \left(q_j \cdot \max_{1 \leqslant i \leqslant m} p_{ij} \right)$, and the expected payoff without experimentation is $EPWE = \max_{1 \leqslant i \leqslant m} \mathbb{E}[A_i] = \max_{1 \leqslant i \leqslant m} \left(\sum_{j=1}^{n} q_j p_{ij} \right)$.

  If EVPI is greater than the experiment cost (EC) then we may conduct the experiment.

- (*EVE - Expected Value of Experimentation*)

  First we determine the expected payoff with experimenation:

  $$EPE = \sum_{k=1}^{t} P(F = f_k) \cdot \max_{1 \leqslant i \leqslant m} (\mathbb{E}[A_i | F = f_k]),$$

  where $P(F = f_k) = \sum_{l=1}^{n} P(F = f_k | S = S_l) \cdot q_l$. Then

  $$EVE = EPE - EPWE.$$

  Its worthwhile to conduct the experiment if this expectation is greater than the experiment cost (EC).

Returning to our example, $EPPI = 0.5 \cdot 15 + 0.5 \cdot 54 = 34.5$, $EPWE = 27 \Rightarrow EVPI = \$7.5$ milion $>$ EC. Using EVPI the cost of the experiment is covered and we may conduct the experiment.

The conditioning probabilities are $P(F = \text{ success } | S = S_1) = 1/4$, $P(F = \text{ success } | S = S_2) = 2/3$, $P(F = \text{ failure } | S = S_1) = 3/4$, $P(F = \text{ failure } | S = S_2) = 1/3$.
We compute first the chances of the survey results:

$$P(\text{success}) = P(F = \text{success}|S = S_1) \cdot 0.5 + P(F = \text{success}|S = S_2) \cdot 0.5 =$$

$$= 1/4 \cdot 0.5 + 2/3 \cdot 0.5 = 11/24,$$

$$P(\text{failure}) = P(F = \text{failure}|S = S_1) \cdot 0.5 + P(F = \text{failure}|S = S_2) \cdot 0.5 =$$

$$3/4 \cdot 0.5 + 1/3 \cdot 0.5 = 13/24.$$

After that we compute the "a posteriori" probabilities:

$$P(S = S_1 | F = \text{ success }) = \frac{P(F = \text{ succes}|S = S_1) \cdot 0.5}{P(\text{success})} = \frac{1/4 \cdot 0.5}{11/24} = 3/11,$$

$$P(S = S_2 | F = \text{ success }) = \frac{P(F = \text{ succes }|S = S_2) \cdot 0.5}{P(\text{success})} = \frac{2/3 \cdot 0.5}{11/24} = 8/11,$$

$$P(S = S_1 | F = \text{failure}) = \frac{P(F = \text{failure}|S = S_1) \cdot 0.5}{P(\text{failure})} = \frac{3/4 \cdot 0.5}{13/24} = 12/13,$$

$$P(S=S_2|F=\text{failure}) = \frac{P(F=\text{failure}|S=S_2)\cdot 0.5}{P(\text{failure})} = \frac{1/3\cdot 0.5}{13/24} = 1/13.$$

The expectations of the conditioning variables are

$$\mathbb{E}[A_1|X=\text{success}] = P(S=S_1|X=\text{success})\cdot 0 + P(S=S_2|X=\text{success})\cdot 54 =$$

$$3/11\cdot 0 + 8/11\cdot 54 = 432/11 \cong 39.28,$$

$$\mathbb{E}[A_1|X=\text{failure}] = P(S=S_1|X=\text{failure})\cdot 0 + P(S=S_2|X=\text{failure })\cdot 54 =$$

$$12/13\cdot 0 + 1/13\cdot 54 = 54/13 \cong 4.16,$$

$$\mathbb{E}[A_2|X=\text{success}] = P(S=S_1|X=\text{success})\cdot 15 + P(S=S_2|X=\text{success})\cdot 54 =$$

$$3/11\cdot 15 + 8/11\cdot 15 = 15,$$

$$\mathbb{E}[A_2|X=\text{failure}] = P(S=S_1|X=\text{failure})\cdot 15 + P(S=S_2|X=\text{failure})\cdot 15 =$$

$$12/13\cdot 15 + 1/13\cdot 15 = 15.$$

Hence,

$$EPE = P(\text{success})\cdot\max\{39.28, 15\} + P(\text{failure})\cdot\max\{4.16, 15\} =$$

$$\frac{11}{24}\cdot 39.28 + \frac{13}{24}\cdot 15 = 26.12 \text{ and}$$

$$EVE = EPE - EPWE = 26.12 - 27 = -0.88 < CE$$

Using EVE, the survey cannot be conducted.

3. Decision Trees.

Suppose that in the former example we have a small variation:

Alternatives:	States of nature:	
	retail guarantee only (S_1)	market succes (S_2)
Produce PC (A_1)	0\$	54 mil.\$
Sell the rights (A_2)	15 mil.\$	15 mil.\$
Probabilities:	0.2	0.8

a) Build and solve the corresponding decision tree.

b) Analyze this decision tree to obtain the optimal policy.

First we compute the chances of the survey results:

$$P(\text{success}) = P(F=\text{success}|S=S_1)\cdot 0.2 + P(F=\text{success}|S=S_2)\cdot 0.8 =$$

$$= 1/4\cdot 0.2 + 2/3\cdot 0.8 = 7/12,$$

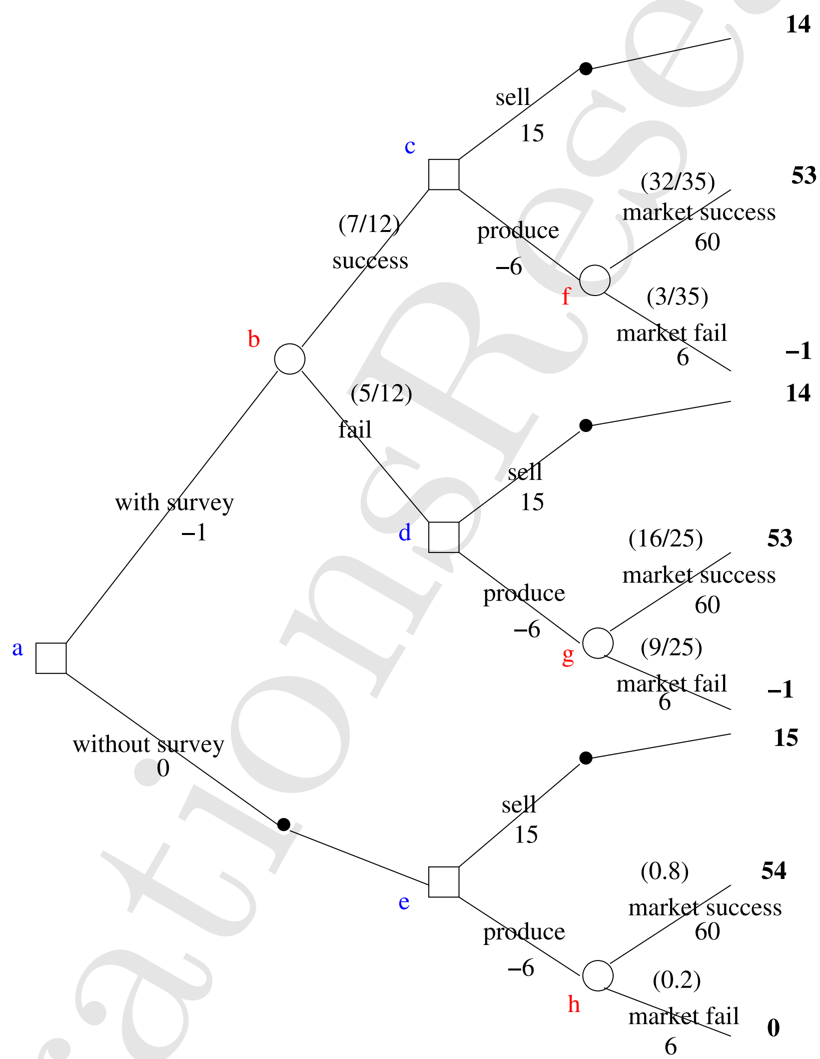
$$P(\text{failure}) = P(F=\text{failure}|S=S_1)\cdot 0.2 + P(F=\text{failure}|S=S_2)\cdot 0.8 =$$

$$3/4\cdot 0.2 + 1/3\cdot 0.8 = 5/12$$

We compute also the "a posteriori" probabilities:

$$P(S=S_1|F=\text{success}) = \frac{P(F=\text{success}|S=S_1)\cdot 0.2}{P(\text{success})} = \frac{1/4\cdot 0.2}{7/12} = 3/35,$$

$$P(S = S_2|F = \text{success}) = \frac{P(F = \text{success}|S = S_2) \cdot 0.8}{P(\text{success})} = \frac{2/3 \cdot 0.8}{7/12} = 32/35,$$
$$P(S = S_1|F = \text{failure}) = \frac{P(F = \text{failure}|S = S_1) \cdot 0.2}{P(\text{failure})} = \frac{3/4 \cdot 0.2}{5/12} = 9/25,$$
$$P(S = S_2|F = \text{failure}) = \frac{P(F = \text{failure } |S = S_2) \cdot 0.8}{P(\text{failure})} = \frac{1/3 \cdot 0.8}{5/12} = 16/25,$$



The initial decision tree.

The analysis of the decision tree:

- For every random node (labeled with ◯: b, f, g, h) we compute the expected payoff.

în f: $\frac{32}{35} \cdot 53 + \frac{3}{35} \cdot (-1) = 48.37$

în g: $\frac{16}{25} \cdot 53 + \frac{9}{25} \cdot (-1) = 33.56$

în h: $\quad 0.8 \cdot 54 + 0.2 \cdot 0 = 43.20$

în b: $\quad \frac{7}{12} \cdot 48.37 + \frac{5}{12} \cdot 33.56 = 42.19$

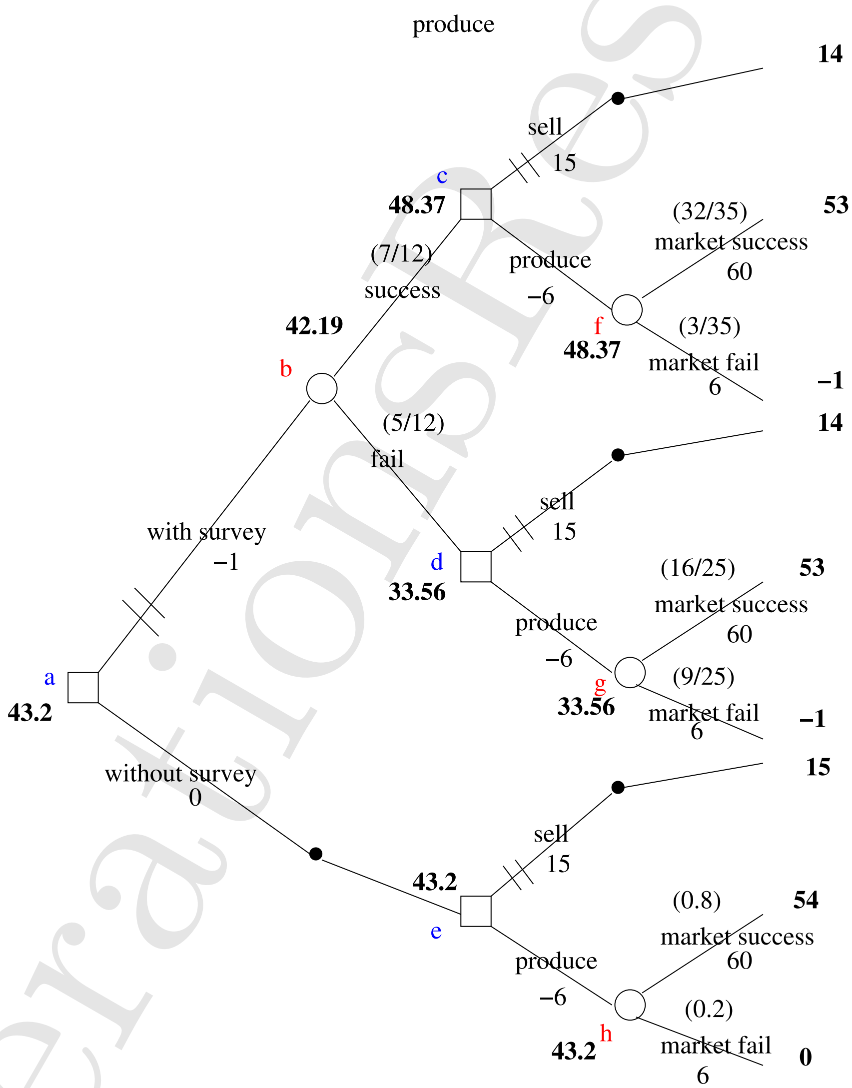
- For every decision node (labeled with cu □: a, c, d, e) we choose the maximum payoff alternative.

in c: $\max\{48.37, 14\} = \mathbf{48.37}$: produce the PC;

in d: $\max\{33.56, 14\} = \mathbf{33.56}$: produce the PC;

in e: $\max\{43.20, 15\} = \mathbf{43.20}$: produce the PC;

in a: $\max\{42.19, 43.20\} = \mathbf{43.20}$: do not conduct the market survey.



The final decision tree.

Optimal policy: Do not conduct the market survey. the PC must be produced. The expected payoff is$43.2 mil.