Homework 2.2

8 points. End term: 7-th week (november 13, 2025)

1. (1 point) Find the dual problems for the following LP problems and check that the dual of the dual is the primal.

$$(\mathbf{a})\begin{cases} \min & z = x_1 - x_2 + 2x_3 - 2x_4 \\ \text{s. t.} & 2x_1 + 3x_2 - x_3 + 2x_4 = 3 \\ & x_1 + 3x_2 + 2x_3 - x_4 \geqslant 2 \\ & -x_1 + 2x_2 - 3x_3 + 2x_4 \leqslant 1 \\ & x_2 \geqslant 0, x_4 \leqslant 0. \end{cases} \qquad (\mathbf{b})\begin{cases} \min & z = 3x_1 - x_2 - 2x_3 + x_4 \\ \text{s. t.} & 2x_1 + x_2 - 2x_3 + 3x_4 \leqslant 3 \\ & -2x_1 + 3x_2 + 2x_3 - 3x_4 \geqslant 1 \\ & x_1 + 2x_2 - x_3 - 2x_4 = -2 \\ & x_1 \geqslant 0, x_2, x_3 \leqslant 0. \end{cases}$$

2. (**2 points**) Consider the following LP problem

$$\begin{cases} \min & z = 2x_1 + 3x_2 + x_3 + 4x_4 \\ \text{s. t.} & x_1 + 2x_2 + x_3 + 2x_4 \geqslant 30 \\ & 2x_1 + x_2 + 3x_3 - 2x_4 \geqslant 20 \\ & 2x_2 + x_4 \geqslant 10 \\ & x_1, x_2, x_3 \geqslant 0 \end{cases}$$

The solution $x_1 = 0, x_2 = 5, x_3 = 20, x_4 = 0$ has been proposed as an optimal solution of the above problem. Build its dual and use complementary slackness in order to check the optimality of this solution.

3. (**2 points**) Consider the following two LP problems

$$(P_1)\begin{cases} \text{minimize} & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \geqslant \mathbf{b} \end{cases} \text{ and } (P_2)\begin{cases} \text{minimize} & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0} \end{cases}.$$

(a) *(1point)* Build the corresponding duals (D_1) and (D_2).

(b) *(1point)* Prove that if (D_1) is unbounded, then (P_2) is infeasible.

4. (**2 points**) Consider an LP problem with a single constraint

$$\begin{array}{ll} \min & z = c_1x_1 + c_2x_2 + \dots c_nx_n \\ \text{s. t.} & a_1x_1 + a_2x_2 + \dots a_nx_n \leqslant b \\ & x_1, x_2, \dots, x_n \geqslant 0 \end{array}$$

Using duality develop a simple rule to determine an optimal solution, if the latter exists.

5. (1 point) Consider the following LP problem

$$(P)\begin{cases} \min & z = x_1 - x_2 + x_3 + 2x_4 \\ \text{s. t.} & x_1 + x_2 + x_3 + 2x_4 \geqslant 3 \\ & -2x_1 + x_2 - 3x_3 - x_4 \leqslant 2 \\ & x_1, x_2, x_3, x_4 \geqslant 0 \end{cases}$$

Use the duality theory to prove that its dual (D) is infeasible - without building the dual.