Exercitiul 7 (b)

Consider the primal problem ($c_1, c_2 \in \mathbb{Z}$) and its dual:

$$(P_1)\left\{\begin{array}{lrl} \max & c_1x_1c_2x_2 & \\ \text{s. t.} & x_1 + x_2 & \leqslant 0 \\ & x_1 - x_2 & \leqslant 0 \\ & x_1 & \leqslant 0 \end{array}\right. \qquad (D_1)\left\{\begin{array}{lrl} \max & 0 & \\ \text{s. t.} & y_1 + y_2 + y_3 & = c_1 \\ & y_1 - y_2 & = c_2 \\ & y_1, y_2, y_3 & \geqslant 0 \end{array}\right.$$

We have to prove that whenever (P_1) is bounded, (D_1) has an integral solution.

Consider $c_1, c_2 \in \mathbb{Z}$ such that (P_1) is bounded.

Observe first that $c_1 + c_2 \geqslant 0$: otherwise taking $x_1^n = -n, x_2^n = -n$ (which is a feasible solution to (P_1)) we have $c_1x_1^n + c_2x_2^n = -n(c_1 + c_2) \to \infty$ when $n \to \infty$, hence (P_1) is not bounded. Second, $c_1 - c_2 \geqslant 0$: otherwise taking $x_1^n = -n, x_2^n = n$ (which is a feasible solution to (P_1)) we have $c_1x_1^n + c_2x_2^n = -n(c_1 - c_2) \to \infty$ when $n \to \infty$, hence (P_1) is not bounded.

Now an integer solution to (D_1) is (any feasible solution to (D_1) is optimal to this problem):

- if $c_1 + c_2$ is odd, then $y_3 = 1$, $y_1 = (c_1 + c_2 - 1)/2 \geqslant 0$, and $y_2 = (c_1 - c_2 - 1)/2 \geqslant 0$.
- if $c_1 + c_2$ is even, then $y_3 = 0$, $y_1 = (c_1 + c_2)/2 \geqslant 0$, and $y_2 = (c_1 - c_2)/2 \geqslant 0$.