Operations Research - Lecture 9

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Table of contents

- The most important development in Linear Optimization Theory, since the emergence of simplex algorithm, are the Interior Point Methods.
- These methods have *better theoretical efficiency* and most of them have also a *better practical performance* than those of the Simplex method.
- The improvement in efficiency was made possible by the fact that Simplex algorithm has an exponential time complexity (see [Klee72]), although in most cases it behaves much more like a polynomial one.
- A polynomial time algorithm has appeared earlier than Karmakar's interior point method([Karmakar84]); this was the ellipsoid method, which had poor performances in practical implementations.

- Karmarkar's algorithm has $\mathcal{O}(n^3 L)$, while the ellipsoid method has $\mathcal{O}((mn^3 + n^4)L)$ time complexity, where n is the number of variables, m the number of constraints, and L is the length of the data.
- A common feature of these methods is that all the intermediate solutions are interior to the polyhedra representing the feasible region, that is the iterated solutions are strictly feasible.
- (A *strictly feasible point* for the set $\{\mathrm{x} \in \mathbb{R}^n_+ : \mathrm{A}\mathrm{x} = \mathrm{b}\}$ is a point x which satisfies $\mathrm{A}\mathrm{x} = \mathrm{b}$ and $\mathrm{x} > 0$.)
- If the simplex algorithm goes along the boundary of the feasible region (more precisely, it moves between its extreme points), the interior point methods iterate through points which are interior (pay attention to the topological sense) to the feasible region.

- Consider the following primal LP problem in standard form (A is a full row rank $m \times n$ matrix)

$$\begin{aligned} \text{minimize} \quad & z = c^T x, \\ \text{subject to} \quad & Ax = b, \\ & x \geqslant 0. \end{aligned} \tag{1}$$

- An iterate x^k satisfies $Ax^k = b$ and $x^k > 0$.
- *Primal methods* computes also a dual solution, convergence is considered reached when this solution is feasible and the duality gap is (almost) zero.

- Now consider the dual problem converted to standard form (s are the slack variables)

$$\begin{aligned} \text{maximize} \quad & w = \mathrm{b}^T \mathrm{y}, \\ \text{subject to} \quad & \mathrm{A}^T \mathrm{y} + \mathrm{s} = \mathrm{c}, \\ & \mathrm{s} \geqslant 0. \end{aligned} \tag{2}$$

- An iterate $\left(\mathrm{y}^k, \mathrm{s}^k\right)$ satisfies $\mathrm{A}^T \mathrm{y}^k + \mathrm{s}^k = \mathrm{c}$ and $\mathrm{s}^k > 0$.
- *Dual methods* computes also a primal solution, convergence is considered reached when this solution is primal feasible and the duality gap is (almost) zero.

- *Primal-dual* methods attempt to solve the primal and dual problems in the same time; x^k and $\left(y^k, s^k\right)$ satisfies exactly the equations of primal and dual problems, respectively, and the non-negativity constraints on x^k and s^k are strictly satisfied.
- Such a primal-dual method ends when the duality gap is zero (to within a certain tolerance).
- If x and (y, s) are feasible for the primal and dual problems, respectively, then the duality gap is

$$c^T x - b^T y = x^T (c - A^T y) = x^T s.$$

- Let $\overline{\mathrm{x}}$ be a feasible solution to the primal problem and $(\overline{\mathrm{y}}, \overline{\mathrm{s}})$ be a feasible solution to the dual problem. These two solutions are be optimal if and only if

$$\overline{\mathrm{x}}^T \overline{\mathrm{s}} = 0 \Leftrightarrow \overline{x}_i \cdot \overline{s}_i = 0, \forall i = \overline{1, n}.$$

- The *primal-dual methods* iterate through a sequence of strictly feasible primal and dual solutions, in each step trying to "reduce" the duality gap, i.e., coming closer to satisfy the slackness conditions.
- At iteration k, the algorithm computes $\mathrm{x}^k(\mu_k), \mathrm{y}^k(\mu_k), \mathrm{s}^k(\mu_k)$, such that, for a certain $\mu_k > 0$ (which decreases with k),

$$\begin{aligned} \mathrm{A}\mathrm{x}^k &= \mathrm{b}, \\ \mathrm{A}^T\mathrm{y}^k + \mathrm{s}^k &= \mathrm{c}, \\ x_i^k s_i^k &= \mu_k, i = \overline{1, n} \\ \mathrm{x}^k, \mathrm{s}^k &\geqslant 0 \end{aligned} \quad (3)$$

- Note that $\mu_k > 0$ forces $x^k > 0$ and $s^k > 0$; on the other hand the duality gap becomes

$$c^T x^k - b^T y^k = \left(x^k\right)^T s^k = n\mu_k.$$

- The algorithm moves through a sequence of primal and dual feasible solutions while decreasing the duality gap.
- If the duality gap were zero, the pair of solutions would be optimal; the stopping condition rely on the size of the duality gap.

- We will use the following notations: $1 = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$. $X = \textit{diag}(x)$ is the diagonal matrix having the elements of x on its diagonal, and, in the same way, we define $S = \textit{diag}(s)$ ($x = X \cdot 1, s = S \cdot 1$, and $XS = \textit{diag}(x_1 s_1, x_2 s_2, \ldots, x_n s_n)$).
- The equations related to complementary slackness may be written as

$$XS1 = \mu 1.$$

- In each iteration the primal-dual algorithm aims to solve the equations (3), although a solution to this system of equations is just approximated.

- Suppose that we have $x^k, s^k > 0$, and y^k satisfying $Ax^k = b$ and $A^T y^k + s^k = c$, but $x_j^k \cdot s_j^k \neq \mu_k$, we shall find new iterates which are closer to satisfying these conditions.
- The new iterates have the form

$$x^{k+1} = x^k + \Delta x^k, s^{k+1} = s^k + \Delta s^k, y^{k+1} = y^k + \Delta y^k.$$

- The new solutions must be feasible, so

$$A\Delta x^k = 0, A^T \Delta y^k + \Delta s^k = 0.$$

- The complementary slackness equations require that

$$(x_j^k + \Delta x_j^k)(s_j^k + \Delta s_j^k) = \mu_k, \text{ or}$$

$$s_j^k \Delta x_j^k + x_j^k \Delta s_j^k + \Delta x_j^k \Delta s_j^k = \mu_k - x_j^k s_j^k, \forall j = \overline{1, n}.$$

- Usually, the differences Δx^k and Δs^k are small, therefore the terms $\Delta x_j^k \Delta s_j^k$ are even smaller. If we ignore them, we get

$$s_j^k \Delta x_j^k + x_j^k \Delta s_j^k = \mu_k - x_j^k s_j^k.$$

- This is the Newton method for linearizing a nonlinear system of equations. $\Delta x, \Delta y$, and Δs are the *Newton directions*.
- At each iteration we solve the following system of linear equations

$$\begin{aligned} S\Delta x + X\Delta s &= \mu 1 - XS1, \\ A\Delta x &= 0, \\ A^T\Delta y + \Delta s &= 0. \end{aligned} \tag{4}$$

- From the last equation we get $\Delta s = -A^T \Delta y$; the first equation can be rewritten as

$$S\Delta x - XA^T \Delta y = \mu 1 - XS1, \text{ or}$$

$$-AS^{-1}XA^T \Delta y = AS^{-1}(\mu 1 - XS1).$$

- Let $D_k = S_k^{-1}X_k$, $v(\mu) = \mu 1 - X_k S_k 1$. The solution to the system (4) is

$$\begin{aligned} \Delta y^k &= -(AD_k A^T)^{-1} AS_k^{-1} v(\mu_k), \\ \Delta s^k &= -A^T \Delta y_k, \\ \Delta x^k &= S_k^{-1} v(\mu_k) - D_k \Delta s_k. \end{aligned} \tag{5}$$

- The algorithm has the following simple description: repeat: given $x^k > 0, y^k$, and $s^k > 0$, compute the Newton directions using (5), update the new iterates

$$x^{k+1} = x^k + \Delta x^k, y^{k+1} = y^k + \Delta y^k, s^{k+1} = s^k + \Delta s^k, \quad (6)$$

and decrease μ_k to μ_{k+1}.

- If we choose $\mu_{k+1} = \theta\mu_k$, where $\theta \in (0, 1)$, and θ is closer enough to 1, then the new iterates will be also strictly feasible.
- It was proved that, in order to achieve strict feasibility in each step, (and, also, the polynomial time complexity of the algorithm: $\mathcal{O}(\sqrt{n}L)$ iterations), we must have $\theta \sim 1 - 3.5/\sqrt{n}$.

- In practical implementations, μ must be decreased more dramatically. However this can have the unwanted effect that the new iterates are not strictly feasible.
- The updating rules (6) are changed to

$$\mathrm{x}^{k+1} = \mathrm{x}^k + \alpha\Delta \mathrm{x}^k, \mathrm{y}^{k+1} = \mathrm{y}^k + \alpha\Delta \mathrm{y}^k, \mathrm{s}^{k+1} = \mathrm{s}^k + \alpha\Delta \mathrm{s}^k, \qquad (7)$$

where the step length, α, is chosen such that the new iterates are strictly feasible.
- Compute first, the distance to the boundary, that is, the largest α which insures

$$x_i^k + \alpha\Delta x_i^k, \mathrm{s}_i^k + \alpha\Delta s_i^k \geqslant 0, i = \overline{1, n}.$$

- The distance to the boundary is

$$\alpha_{max} = \min\{\alpha_x, \alpha_s\}, \text{ where}$$

$$\alpha_x = \min\{-x_i^k/\Delta x_i^k : \Delta x_i^k < 0\}, \alpha_s = \min\{-s_i^k/\Delta s_i^k : \Delta s_i^k < 0\},$$

- The step length, α, used in updating the new iterates is a fraction of α_{max}, say

$$\alpha = 0.99999\alpha_{max}.$$

Let $x^0 > 0, y^0, s^0 > 0$ be strictly feasible initial estimates of primal, dual, and dual slack variables, $k = 0$, $\mu_0 > 0$.

The Gap Test. If $\left|\left(x^k\right)^T \cdot s^k\right| \leqslant 10^{-p}$, then stop, we have an estimate of an optimal pair of solutions.

The Main Step. Compute the Newton distances using the system of equations (5); compute α. $k++$.

The Update Update the new iterates x^k, y^k, s^k using (7) and go to the gap test.

- Consider the following LP problem

$$\begin{aligned} \text{minimize} \quad & z = -x_1 - 2x_2 \\ \text{subject to} \quad & -2x_1 + x_2 \leqslant 2 \\ & -x_1 + 2x_2 \leqslant 7 \\ & x_1 + 2x_2 \leqslant 3 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

- We convert this problem to the standard form

$$\begin{aligned} \text{minimize} \quad & z = -x_1 - 2x_2 \\ \text{subject to} \quad & -2x_1 + x_2 + x_3 = 2 \\ & -x_1 + 2x_2 + x_4 = 7 \\ & x_1 + 2x_2 + x_5 = 3 \\ & x_1, x_2, \ldots, x_5 \geqslant 0 \end{aligned} \tag{8}$$

- The dual of problem (8) is

$$\begin{array}{lrcr} \text{maximize} & w = 2y_1 + 7y_2 + 3y_3 & & \\ \text{subject to} & -2y_1 - y_2 + y_3 & \leqslant & -1 \\ & y_1 + 2y_2 + 2y_3 & \leqslant & -2 \\ & y_1 & \leqslant & 0 \\ & y_2 & \leqslant & 0 \\ & y_3 & \leqslant & 0 \end{array}$$

- In standard form the dual is

$$\begin{array}{lrcr} \text{maximize} & w = 2y_1 + 7y_2 + 3y_3 & & \\ \text{subject to} & -2y_1 - y_2 + y_3 + s_1 & = & -1 \\ & y_1 + 2y_2 + 2y_3 + s_2 & = & -2 \\ & y_1 + s_3 & = & 0 \\ & y_2 + s_4 & = & 0 \\ & y_3 + s_5 & = & 0 \\ & s_1, s_2, \ldots, s_5 \geqslant 0 & & \end{array} \tag{9}$$

- An initial pair of strictly feasible solutions to problems (8) and (9) is

$$x^0 = (0.5\ 0.5\ 2.5\ 6.5\ 1.5)^T,$$

$$y^0 = (-1\ -1\ -5)^T, s^0 = (1\ 11\ 1\ 1\ 5)^T.$$

- We also have

$$X_0 = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 2.5 & 0 & 0 \\ 0 & 0 & 0 & 6.5 & 0 \\ 0 & 0 & 0 & 0 & 1.5 \end{bmatrix}, S_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

- Let $\mu_0 = 10$. At first iteration we get

$$D_0 = \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/22 & 0 & 0 & 0 \\ 0 & 0 & 5/2 & 0 & 0 \\ 0 & 0 & 0 & 13/2 & 0 \\ 0 & 0 & 0 & 0 & 3/10 \end{bmatrix}, v(\mu_0) = \begin{bmatrix} 9.5 \\ 4.5 \\ 7.5 \\ 3.5 \\ 2.5 \end{bmatrix},$$

$$AD_0A^T = \begin{bmatrix} 4.5455 & 1.0909 & -0.9091 \\ 1.0909 & 7.1818 & -0.3182 \\ -0.9091 & -0.3182 & 0.9818 \end{bmatrix},$$

$$(AD_0A^T)^{-1} = \begin{bmatrix} 0.2767 & -0.0311 & 0.2461 \\ -0.0311 & 0.1448 & 0.1081 \\ 0.2461 & 0.1081 & 1.2523 \end{bmatrix}$$

- The Newton directions are

$$\Delta x^0 = \begin{bmatrix} 3.7885 \\ -0.5356 \\ 8.1126 \\ 4.8598 \\ -2.7172 \end{bmatrix}, \Delta s^0 = \begin{bmatrix} 11.4231 \\ 20.7843 \\ -0.2450 \\ -0.2092 \\ 10.7239 \end{bmatrix}, \Delta y^0 = \begin{bmatrix} 0.2450 \\ 0.2092 \\ -10.7239 \end{bmatrix}.$$

- We compute the distance to the boundary

$$\alpha_{max} = 0.552 \left(\text{as } \alpha_x = -\frac{x_5^0}{\Delta x_5^0} = 0.552 \text{ and } \alpha_s = -\frac{s_3^0}{\Delta s_3^0} = 4.0812 \right).$$

- $\alpha = 0.99999 \cdot \alpha_{max} \cong 0.552.$

- The new iterates are

$$x^1 = \begin{bmatrix} 2.5914 \\ 0.2043 \\ 6.9785 \\ 9.1828 \\ -0.0000 \end{bmatrix}, s^1 = \begin{bmatrix} 7.3060 \\ 22.4738 \\ 0.8647 \\ 0.8845 \\ 10.9200 \end{bmatrix}, y^1 = \begin{bmatrix} -0.8647 \\ -0.8845 \\ -10.9200 \end{bmatrix}.$$

- With these values, the duality gap is $\left(x^1\right)^T \cdot s^1 = 37.6812$, and the complementary slackness conditions have the residual $\left(x^1\right)^T \cdot s^1 - n\mu = -12.3188$.
- We choose $\theta = 0.1$.

- After some more iterations the situation becomes

k	μ	$\left(\mathrm{x}^k\right)^T \cdot \mathrm{s}^k - n\mu$	$\left(\mathrm{x}^k\right)^T \cdot \mathrm{s}^k$
1	10^1	-1×10^1	4×10^1
2	10^0	-4×10^0	1×10^0
3	10^{-1}	-2×10^{-1}	1×10^{-1}
4	10^{-2}	-2×10^{-2}	1×10^{-2}
5	10^{-3}	-3×10^{-3}	1×10^{-3}
6	10^{-4}	-2×10^{-4}	1×10^{-4}
7	10^{-5}	-3×10^{-5}	1×10^{-5}
8	10^{-6}	-2×10^{-6}	1×10^{-6}
9	10^{-7}	-3×10^{-7}	1×10^{-7}

- The current iterates are

$$x^9 = \begin{bmatrix} 2.1794 \\ 0.4103 \\ 5.9486 \\ 8.3588 \\ 2 \times 10^{-8} \end{bmatrix}, s^9 = \begin{bmatrix} 5 \times 10^{-8} \\ 1 \times 10^{-7} \\ 2 \times 10^{-12} \\ 4 \times 10^{-9} \\ 1.0000 \end{bmatrix}, y^9 = \begin{bmatrix} -2 \times 10^{-12} \\ -4 \times 10^{-9} \\ -1.0000 \end{bmatrix}.$$

- The objective values are $z = c^T x_9 = -3.0000 = b^T y_9 = w$.
- The true solution to dual problem is $s_* = (0\,0\,0\,0\,1)^T, y_* = (0\,0\,-1)^T$.
- All the points on the line segment between $x^1 = (0\ 1.5\ 0.5\ 4\ 0)^T$ and $x^2 = (3\ 0\ 8\ 10\ 0)^T$ are optimal solutions to the primal problem. One of these points is

$$x_* = (2.1794\ \ 0.4103\ \ 5.9486\ \ 8.3588\ \ 0)^T$$

- The above algorithm has, in the initial iteration, a pair of strictly feasible solutions for the primal and dual problems. That is

$$Ax = b, A^T y + s = c, \text{ with } x > 0, s > 0.$$

- It may be difficult to find such solutions, therefore it is useful to accommodate the primal-dual algorithm so that initial infeasible solutions can be used.
- Suppose we have $x, s > 0$ and y. We try to find new iterates

$$A(x+\Delta x) = b, A^T(y+\Delta y)+(s+\Delta s) = c, (x_i+\Delta x_i)(s_i+\Delta s_i) = \mu, \forall i.$$

- Again, if we ignore the terms $(\Delta x_i \Delta s_i)$, then these equations become

$$\begin{aligned} S\Delta x + X\Delta s &= \mu 1 - XS1 = v(\mu), \\ A\Delta x &= b - Ax = \rho_P, \\ A^T\Delta y + \Delta s &= c - A^T y - s = \rho_D. \end{aligned}$$

- Where ρ_P is the residual for the primal constraints $\mathrm{A}\mathrm{x} = \mathrm{b}$, and ρ_D is the residual to the dual constraints $\mathrm{A}^T\mathrm{y} + \mathrm{s} = \mathrm{c}$.
- Finally, after a similar analysis, we get the following system of linear equations which gives us the Newton directions:

$$
\begin{aligned}
\Delta \mathrm{y}_k &= -(\mathrm{A}\mathrm{D}_k\mathrm{A}^T)^{-1}\left[\mathrm{A}\mathrm{S}_k^{-1}v(\mu_k) - \mathrm{A}\mathrm{D}_k\rho_D^k - \rho_P^k\right], \\
\Delta \mathrm{s}_k &= -\mathrm{A}^T\Delta \mathrm{y}_k + \rho_D^k, \\
\Delta \mathrm{x}_k &= \mathrm{S}_k^{-1}v(\mu_k) - \mathrm{D}_k\Delta \mathrm{s}_k.
\end{aligned}
\tag{10}
$$

- Therefore, in the step of the primal-dual algorithm we must replace (5) by (10).
- However with this modification the method may fail to converge if the primal or dual problem has not feasible solutions.

- The already described primal-dual algorithm computes the Newton directions under the assumption that Δx and Δs have both small norms.
- These assumptions are not always true, especially when the current iterates are far from the optimal ones.
- If we keep the second terms in our equations we get the following system

$$\begin{aligned} S\Delta x + X\Delta s &= \mu 1 - XS1 - \Delta X \Delta S 1, \\ A\Delta x &= b - Ax, \\ A^T\Delta y + \Delta s &= c - A^T y - s, \end{aligned} \tag{11}$$

where $\Delta X = diag(\Delta x_i)$ and $\Delta S = diag(\Delta s_i)$

The Predictor-Corrector Algorithm

- The system (11) is solved in two steps: a *predictor* step and a *corrector* one.
- In the predictor step we get a prediction for Δx, Δs, and a prediction for μ.
- In the corrector step the predicted values are used to find a solution to the system (11).

- The predictor step attempts to solve the system

$$\begin{aligned} S\Delta x + X\Delta s &= -XS1, \\ A\Delta x &= b - Ax, \\ A^T\Delta y + \Delta s &= c - A^T y - s, \end{aligned} \tag{12}$$

which is the former system from where the term $(\mu 1 - \Delta X \Delta S 1)$ vanished.

- After solving this system we get $\Delta\widehat{X}$ and $\Delta\widehat{S}$, and, from here, $\widehat{x}$ and $\widehat{s}$, which are used for updating $\mu = \widehat{x}^T\widehat{s}$.
- In the corrector step we will solve

$$\begin{aligned} S\Delta x + X\Delta s &= \widehat{\mu}1 - XS1 - \Delta\widehat{X}\Delta\widehat{S}1, \\ A\Delta x &= b - Ax, \\ A^T\Delta y + \Delta s &= c - A^T y - s, \end{aligned} \tag{13}$$

and find the desired Newton directions $\Delta x, \Delta y$, and Δs.

- From the third equation in (12) we get $\Delta s = -A^T \Delta y + c - A^T y - s$, and from the first and the second equations

$$b - Ax = -AS^{-1}XA^T \Delta y - AS^{-1}c +$$

$$+AS^{-1}A^T y - AS^{-1}XS1.$$

- From here we can obtain (with $D = S^{-1}X$):

$$\Delta y = (ADA^T)^{-1} \left[Ax - b - AS^{-1}c + \right.$$

$$\left. +AS^{-1}A^T y - AS^{-1}XS1\right],$$

$$\Delta s = -A^T \Delta y + c - A^T y - s, \tag{14}$$

$$\Delta x = -S^{-1}X\Delta sXS1.$$

The Predictor Step

- This was the predictor step.
- From here we can compute

$$
\begin{aligned}
\widehat{\mathrm{x}}^k &= \mathrm{x}^k + \Delta\widehat{\mathrm{x}}^k,\\
\widehat{\mathrm{s}}^k &= \mathrm{s}^k + \Delta\widehat{\mathrm{s}}^k,\\
\Delta\widehat{\mathrm{X}}_k &= diag(\Delta\widehat{\mathrm{x}}^k),\\
\Delta\widehat{\mathrm{S}}_k &= diag(\Delta\widehat{\mathrm{s}}^k),\\
\widehat{\mu}_k &= \left(\widehat{\mathrm{x}}^k\right)^T\widehat{\mathrm{s}}^k.
\end{aligned} \tag{15}
$$

- After the predictions were made the system for the corrector step, (13), becomes

$$\begin{aligned} S\Delta x + X\Delta s &= \widehat{\mu}1 - XS1 - \Delta\widehat{X}\Delta\widehat{S}1, \\ A\Delta x &= b - Ax, \\ A^T\Delta y + \Delta s &= c - A^T y - s, \end{aligned}$$

- We get from the third equation $\Delta s = c - A^T y - s - A^T \Delta y$. Then, from the first equation, we obtain

$$S\Delta x + Xc - XA^T y - Xs - XA^T\Delta y = \widehat{\mu}1 - XS1 - \Delta\widehat{X}\Delta\widehat{S}1.$$

- Multiplying this equation by AS^{-1} and using the second equation, we get

$$b - Ax + AS^{-1}Xc - AS^{-1}XA^T y - AS^{-1}Xs - AS^{-1}XA^T\Delta y =$$

$$= AS^{-1}\widehat{\mu}1 - AS^{-1}XS1 - AS^{-1}\Delta\widehat{X}\Delta\widehat{S}1.$$

- If $D = S^{-1}X$, then

$$\Delta y = (ADA^T)^{-1}(b - Ax + AS^{-1}Xc) - y -$$
$$-(ADA^T)^{-1}(AS^{-1}\hat{\mu}1 - AS^{-1}XS1 - AS^{-1}\Delta\hat{X}\Delta\hat{S}1)$$

- Or

$$\Delta y = (ADA^T)^{-1}(b - Ax) - y -$$
$$-(A^T)^{-1}(X^{-1}\hat{\mu}1 - S1 - c - X^{-1}\Delta\hat{X}\Delta\hat{S}1) =$$
$$= (ADA^T)^{-1}(b - Ax) +$$
$$+(A^T)^{-1}\left[S1 + c - X^{-1}\left(\hat{\mu}1 - \Delta\hat{X}\Delta\hat{S}1\right)\right] - y$$

- The corrector step gives us

$$\begin{aligned}
\Delta y^k &= (AD_kA^T)^{-1}(b - Ax_k) - y^k + \\
&+ (A^T)^{-1}\left[S_k 1 + c - X_k^{-1}\left(\widehat{\mu}_k - \Delta\widehat{X}_k\Delta\widehat{S}_k\right)1\right], \\
\Delta s^k &= c - A^Ty^k - s^k - A^T\Delta y^k, \\
\Delta x^k &= S_k^{-1}\left(\widehat{\mu}_k - \Delta\widehat{X}_k\Delta\widehat{S}_k\right)1 - D_kS_k1 - D_k\Delta s^k.
\end{aligned} \tag{16}$$

Let $x^0 > 0, y^0, s^0 > 0$ be strictly feasible initial estimates of primal, dual, and dual slack variables, $k = 0$, $\mu_0 > 0$

The Gap Test. If $\left(\left|\left(x^k\right)^T s^k\right| \leqslant 10^{-p}\right)$, then stop, we have an estimate of an optimal pair of solutions.

The Predictor Step. Compute the the predictions of $\Delta\widehat{X}_k$, $\Delta\widehat{S}_k$, and $\widehat{\mu}_k$, using (14) and (15).

The Corrector Step. Compute the Newton directions using (16); compute α. $k++$.

The Update. Update the new iterates x^k, y^k, s^k using (7) and go to the gap test.

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