# Operations Research - Lecture 7

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- Let us consider again the ILP, MILP, and BILP problems

$$\begin{aligned} \text{maximize} \quad & z = \mathrm{c}^T \mathrm{x}, \\ \text{subject to} \quad & \mathrm{A}\mathrm{x} \leqslant \mathrm{b}, \\ & \mathrm{x} \in \mathbb{Z}_+^n. \end{aligned} \tag{1}$$

$$\begin{aligned} \text{maximize} \quad & z = \mathrm{c}^T \mathrm{x}, \\ \text{subject to} \quad & \mathrm{A}\mathrm{x} \leqslant \mathrm{b}, \\ & x_i \in \mathbb{Z}_+, \forall i \in \mathcal{I}. \end{aligned} \tag{2}$$

where $\varnothing \neq \mathcal{I} \subsetneq \{1, 2, \ldots, n\}$.

$$\begin{aligned} \text{maximize} \quad & z = \mathrm{c}^T \mathrm{x}, \\ \text{subject to} \quad & \mathrm{A}\mathrm{x} \leqslant \mathrm{b}, \\ & \mathrm{x} \in \{0, 1\}^n. \end{aligned} \tag{3}$$

- Any ILP problem admits a *natural relaxation* obtained by discarding the integrality requirements on x.
- Problems (1) and (2) have the same relaxation:

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^T \mathbf{x}, \\ \text{subject to} \quad & A\mathbf{x} \leqslant \mathbf{b}, \\ & \mathbf{x} \in \mathbb{R}_+^n. \end{aligned} \tag{4}$$

- Problem (3) has a specific relaxation:

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^T \mathbf{x}, \\ \text{subject to} \quad & A\mathbf{x} \leqslant \mathbf{b}, \\ & \mathbf{x} \in [0, 1]^n. \end{aligned} \tag{5}$$

- Linear relaxations of ILP problems usually have greater objective function value because the feasible region of such a relaxation contains the feasible region of the ILP problem.
- If for solving the relaxation we already have theoretical tools such as Simplex algorithm, ILP problems are much harder to solve.
- An optimal solution of the relaxation might not be a feasible solution to the ILP original problem, due to the integrality restrictions.
- There are situations when rounding an optimal solution for the relaxation can give an optimal/sub-optimal solution to the original ILP problem.

- For the sake of simplicity of our exposition we will analyze only the MILP problem in the following form

$$\begin{aligned} \text{maximize} \quad & z = \mathrm{c}^T\mathrm{x} + \mathrm{h}^T\mathrm{y}, \\ \text{subject to} \quad & \mathrm{A}\mathrm{x} + \mathrm{G}\mathrm{y} \leqslant \mathrm{b}, \\ & \mathrm{x} \in \mathbb{Z}_+^n, \mathrm{y} \in \mathbb{R}_+^p. \end{aligned} \tag{6}$$

- Or, equivalently,

$$(MILP) \ \max\{\mathrm{c}^T\mathrm{x} + \mathrm{h}^T\mathrm{y} \ : \ (\mathrm{x},\mathrm{y}) \in \mathcal{P}\}, \tag{7}$$

where

$$\mathcal{P} = \{(\mathrm{x},\mathrm{y}) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p \ : \ \mathrm{A}\mathrm{x} + \mathrm{G}\mathrm{y} \leqslant \mathrm{b}\}$$

- The *natural relaxation* of the set $\mathcal{P}$ is

$$\mathcal{R} = \{(x, y) \in \mathbb{R}_+^{n+p} : Ax + Gy \leqslant b\}.$$

- The *natural relaxation* of problem (7) is

$$(LP)\ \max\{c^T x + h^T y : (x, y) \in \mathcal{R}\}, \quad (8)$$

- We will assume throughout this section that problem (7) has finite optimum. Let $(x^*, y^*)$ be an optimal solution and $z_*$ be the optimal value of this problem.
- The assumption from above implies that problem (8) has an optimal rational solution $(x^0, y^0)$ with optimal value $z_0$.
- This result comes without a proof and is based on the rationality of the input data (why?). An optimal solution is in fact always a rational one (why?).

- Suppose that $(x^0, y^0)$ and $z_0$ can be obtained by using an LP solver (using for example the simplex algorithm).
- Since $\mathcal{P} \subseteq \mathcal{R}$, we must have $z^* \leqslant z_0$. If $x^0 \in \mathbb{Z}^n$, then $(x^0, y^0) \in \mathcal{P}$ and therefore $z_* = z_0$ - in this case problem (6) is solved.
- What happens when $x^0 \notin \mathbb{Z}^n$, that is, at least a component of $x^0$ is fractional? The possible answers give different strategies for solving problem (6).
- We will describe two distinct strategies: the *Branch-and-Bound Method* and the *Cutting Plane Method*.

- Suppose that $x_j^0 \notin \mathbb{Z}$, then we define

$$\mathcal{P}_1 = \mathcal{P} \cap \{(\mathrm{x}, \mathrm{y}) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : x_j \leqslant \lfloor x_j^0 \rfloor\},$$

$$\mathcal{P}_2 = \mathcal{P} \cap \{(\mathrm{x}, \mathrm{y}) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : x_j \geqslant \lceil x_j^0 \rceil\}.$$

- Obviously, $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ and $\mathcal{P}_1 \cap \mathcal{P}_2 = \varnothing$.
- Consider two new MILP problems

$$(MILP_1)\ \max\{\mathrm{c}^T\mathrm{x} + \mathrm{h}^T\mathrm{y} : (\mathrm{x}, \mathrm{y}) \in \mathcal{P}_1\}, \quad (9)$$

$$(MILP_2)\ \max\{\mathrm{c}^T\mathrm{x} + \mathrm{h}^T\mathrm{y} : (\mathrm{x}, \mathrm{y}) \in \mathcal{P}_2\}. \quad (10)$$

- The optimal solution of (6) is the best among the optimal solutions of (9) and (10). The idea of branching comes here: by solving the two new sub-problems we will solve the original problem.

- Let the natural linear relaxations of $\mathcal{P}_1$ and $\mathcal{P}_2$ be

$$\mathcal{R}_1 = \mathcal{R} \cap \{(\mathrm{x}, \mathrm{y}) \in \mathbb{R}_+^{n+p} \; : \; x_j \leqslant \lfloor x_j^0 \rfloor\},$$

$$\mathcal{R}_2 = \mathcal{R} \cap \{(\mathrm{x}, \mathrm{y}) \in \mathbb{R}_+^{n+p} \; : \; x_j \geqslant \lceil x_j^0 \rceil\}.$$

- The natural relaxation corresponding problems are

$$(LP_1) \; \max \{\mathrm{c}^T \mathrm{x} + \mathrm{h}^T \mathrm{y} \; : \; (\mathrm{x}, \mathrm{y}) \in \mathcal{R}_1\}, \tag{11}$$

$$(LP_2) \; \max \{\mathrm{c}^T \mathrm{x} + \mathrm{h}^T \mathrm{y} \; : \; (\mathrm{x}, \mathrm{y}) \in \mathcal{R}_2\}. \tag{12}$$

- Concerning each of these two sub-problems we have two possible situations.

1. Obviously, if $\mathcal{R}_i = \varnothing$ (that is, problem $(LP_i)$ is infeasible), then $\mathcal{P}_i = \varnothing$ and $(MILP_i)$ is also infeasible. We say that this problem is *fathomed* or *pruned by infeasibility*.
2. Otherwise, let $(\mathrm{x}^i, \mathrm{y}^i)$ be an optimal solution of $(LP_i)$ and $z_i$ be its optimal value.
   2.1 If $\mathrm{x}^i \in \mathbb{Z}^n$, then $(\mathrm{x}^i, \mathrm{y}^i)$ is an optimal solution of $(MILP_i)$ and $z_i \leqslant z_*$. We say that $(MILP_i)$ is *fathomed* or *pruned by integrality*.
   2.2 If $\mathrm{x}^i \notin \mathbb{Z}^n$, and $z_i$ is smaller (or equal) than the best lower bound of $\mathrm{z}_*$, then $(LP_i)$ doesn't have a better solution than the best current solution and the problem is *fathomed* or *pruned by bound*.

2.3 If $x^i \notin \mathbb{Z}^n$, and $z_i$ is greater than the best lower bound of $z_*$, then $(MILP_i)$ can contain an optimal solution of $(MILP)$. Let $x_h^i$ a non-integral component of $x^i$. We define the polyhedra

$$\mathcal{P}_{i1} = \mathcal{P}_i \cap \{(x, y) \in \mathbb{R}^{n+p} \,:\, x_h \leqslant \lfloor x_h^i \rfloor\},$$

$$\mathcal{P}_{i2} = \mathcal{P}_i \cap \{(x, y) \in \mathbb{R}^{n+p} \,:\, x_h \geqslant \lceil x_h^i \rceil\}.$$

We build two new sub-problems $(MILP_{i1})$ and $(MILP_{i2})$; after that we iterate the process.

- Consider the following ILP problem and its LP relaxation

$$(P)\begin{cases} \max & z = 17x_1 + 12x_2 \\ \text{s. t.} & 10x_1 + 7x_2 \leqslant 40 \\ & x_1 + 2x_2 \leqslant 5 \\ & x_1, x_2 \geqslant 0 \\ & x_1, x_2 \in \mathbb{Z} \end{cases} \quad (R)\begin{cases} \max & z = 17x_1 + 12x_2 \\ \text{s. t.} & 10x_1 + 7x_2 \leqslant 40 \\ & x_1 + 2x_2 \leqslant 5 \\ & x_1, x_2 \geqslant 0 \end{cases}$$

- An optimal solution to the natural LP relaxation, $(\mathcal{R})$, is $\mathrm{x} = (x_1, x_2) = (1.67, 3.33)^T$, and the optimal value is $z = 205/3 \approx 68.33$.
- By branching on variable $x_1$ we create two new ILP sub-problems $(P_1)$ and $(P_2)$, those relaxations are $(R_1)$ and $(R_2)$, respectively.

$$(R_1)\begin{cases} \max & z = 17x_1 + 12x_2 \\ \text{s. t.} & 10x_1 + 7x_2 \leqslant 40 \\ & x_1 + 2x_2 \leqslant 5 \\ & x_1 \leqslant 1 \\ & x_1, x_2 \geqslant 0 \end{cases} \qquad (R_2)\begin{cases} \max & z = 17x_1 + 12x_2 \\ \text{s. t.} & 10x_1 + 7x_2 \leqslant 40 \\ & x_1 + 2x_2 \leqslant 5 \\ & x_1 \geqslant 2 \\ & x_1, x_2 \geqslant 0 \end{cases}$$

- The problem $(R_1)$ has the optimal solution $\mathrm{x} = (1,3)^T$, and the optimal value $z = 65$; this problem is pruned by integrality.
- $z = 65$ becomes the new best lower bound for the optimal value of original problem $(P)$ and the current best solution becomes $\mathrm{x} = (1,3)^T$.
- The problem $(R_2)$ has the optimal solution $\mathrm{x} = (2, 2.86)^T$, and optimal value $z = 68.29$; this solution is not feasible for the ILP problem $(P_2)$.

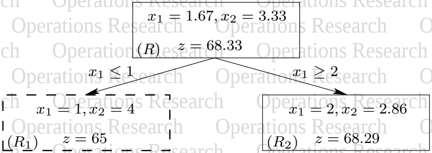


Figure: The enumeration tree after solving $(R_2)$

- Since the optimal value of problem $(R_2)$ is better than the current best optimal value of an integer solution, we create two new subproblems $(P_3)$ and $(P_4)$, by branching on the variable $x_2$: one with the additional constraint $x_2 \leqslant 2$, the other with $x_2 \geqslant 3$.

$$(R_3)\begin{cases} \max & z = 17x_1 + 12x_2 \\ \text{s. t.} & 10x_1 + 7x_2 \leqslant 40 \\ & x_1 + 2x_2 \leqslant 5 \\ & x_1 \geqslant 2 \\ & x_2 \leqslant 2 \\ & x_1, x_2 \geqslant 0 \end{cases} \qquad (R_4)\begin{cases} \max & z = 17x_1 + 12x_2 \\ \text{s. t.} & 10x_1 + 7x_2 \leqslant 40 \\ & x_1 + 2x_2 \leqslant 5 \\ & x_1 \geqslant 2 \\ & x_2 \geqslant 3 \\ & x_1, x_2 \geqslant 0 \end{cases}$$

- The problem $(R_3)$ has the optimal solution $\mathrm{x} = (2.6, 2)^T$, and the optimal value $z = 68.2$; this solution is not feasible for the ILP problem $(P_3)$.
- The problem $(R_4)$ has not feasible solutions therefore is pruned by infeasibility.

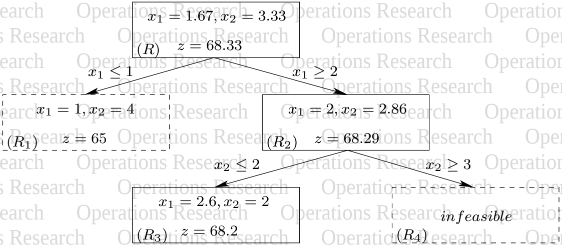


Figure: The enumeration tree after solving $(R_4)$

- Since the optimal value of problem $(R_3)$ is better than the current best optimal value of an integer solution, we create two new subproblems $(P_5)$ and $(P_6)$, by branching on the variable $x_1$: one with the additional constraint $x_1 \leqslant 2$, the other with $x_1 \geqslant 3$.

$$(R_5)\begin{cases} \max & z = 17x_1 + 12x_2 \\ \text{s. t.} & 10x_1 + 7x_2 \leqslant 40 \\ & x_1 + 2x_2 \leqslant 5 \\ & x_1 = 2 \\ & x_2 \leqslant 2 \\ & x_1, x_2 \geqslant 0 \end{cases} \qquad (R_6)\begin{cases} \max & z = 17x_1 + 12x_2 \\ \text{s. t.} & 10x_1 + 7x_2 \leqslant 40 \\ & x_1 + 2x_2 \leqslant 5 \\ & x_2 \leqslant 2 \\ & x_1 \geqslant 3 \\ & x_1, x_2 \geqslant 0 \end{cases}$$

- The problem $(R_5)$ has the optimal solution $x = (2, 2)^T$, and the optimal value $z = 58$; this problem is pruned by integrality but its optimal value is no better than the current best optimal value which is 65.
- The problem $(R_6)$ has the optimal solution $x = (3, 1.43)^T$, and the optimal value $z = 68.14$; this solution is not feasible for the ILP problem $(P_6)$.

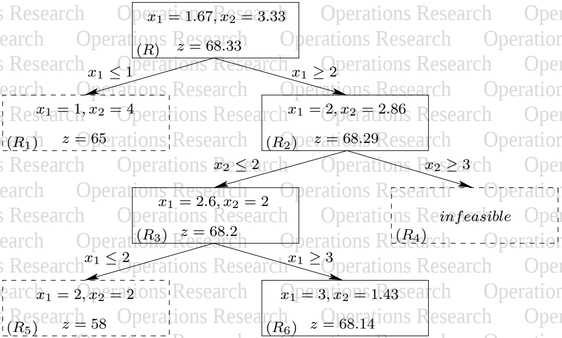


Figure: The enumeration tree after solving $(R_6)$

- Since the optimal value of problem $(R_6)$ is better than the value of the best-so-far integer solution, we create the sub-problems $(P_7)$ and $(P_8)$: one with the additional constraint $x_2 \leqslant 1$, the other with $x_2 \geqslant 2$.

$$(R_7)\begin{cases} \max & z = 17x_1 + 12x_2 \\ \text{s. t.} & 10x_1 + 7x_2 \leqslant 40 \\ & x_1 + 2x_2 \leqslant 5 \\ & x_2 \leqslant 1 \\ & x_1 \geqslant 3 \\ & x_1, x_2 \geqslant 0 \end{cases} \qquad (R_8)\begin{cases} \max & z = 17x_1 + 12x_2 \\ \text{s. t.} & 10x_1 + 7x_2 \leqslant 40 \\ & x_1 + 2x_2 \leqslant 5 \\ & x_2 = 2 \\ & x_1 \geqslant 3 \\ & x_1, x_2 \geqslant 0 \end{cases}$$

- The problem $(R_7)$ has the optimal solution $x = (3.3, 1)^T$, and the optimal value $z = 68.1$; this solution is not feasible for the ILP problem $(P_7)$.
- The problem $(R_8)$ has not feasible solutions therefore is pruned by infeasibility.

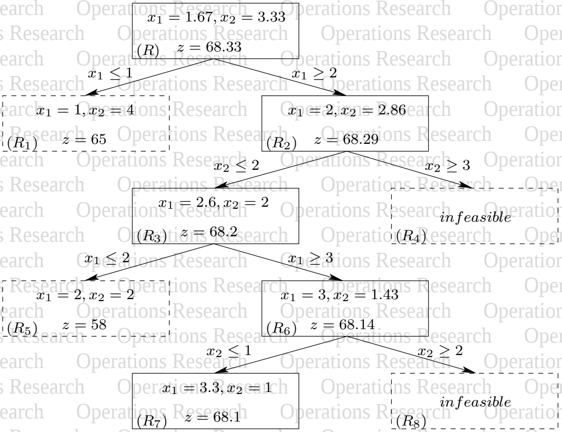


Figure: The enumeration tree after solving $(R_8)$

- Since the optimal value of problem $(R_7)$ is better than the current best optimal value of an integer solution, and its solution is fractional, we create two new sub-problems $(P_9)$ and $(P_{10})$, by branching on the variable $x_1$: one with the additional constraint $x_1 \leqslant 3$, the other with $x_1 \geqslant 4$.

$$(R_9)\begin{cases} \max & z = 17x_1 + 12x_2 \\ \text{s. t.} & 10x_1 + 7x_2 \leqslant 40 \\ & x_1 + 2x_2 \leqslant 5 \\ & x_2 \leqslant 1 \\ & x_1 = 3 \\ & x_1, x_2 \geqslant 0 \end{cases} \qquad (R_{10})\begin{cases} \max & z = 17x_1 + 12x_2 \\ \text{s. t.} & 10x_1 + 7x_2 \leqslant 40 \\ & x_1 + 2x_2 \leqslant 5 \\ & x_2 \leqslant 1 \\ & x_1 \geqslant 4 \\ & x_1, x_2 \geqslant 0 \end{cases}$$

- The problem $(R_9)$ has the optimal solution $\mathrm{x} = (3,1)^T$, and the optimal value $z = 63$; this problem is pruned by integrality but its optimal value is no better than the current best optimal value which is 65.
- The problem $(R_{10})$ has the optimal solution $\mathrm{x} = (4,0)^T$, and the optimal value $z = 68$; this problem is pruned by integrality and its optimal value is better than the current best optimal value which is 65.
- $z = 68$ becomes the new best lower bound for the optimal value of original problem $(P)$ and the current best solution becomes $\mathrm{x} = (4,0)^T$.

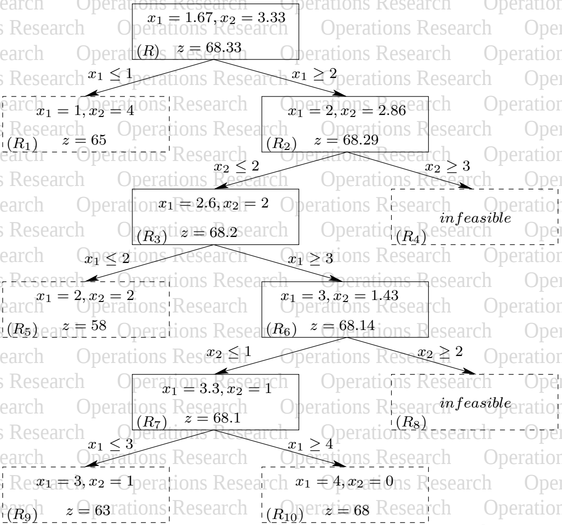


Figure: The final enumeration tree.

- We can observe from the above example that we have to traverse a binary tree, having on its nodes LP problems.
- After we solve such a problem, the corresponding node becomes a leaf (when the problem is pruned) or it expands creating two new children.
- Usually the traversal strategy is *depth-first-search*.
- We will shortly discuss the reasons for using *dfs* strategy in building and traversing the enumeration tree.

# Notes Regarding the Algorithm



- Empirical knowledge shows that most integer solutions of the original problem lie deep in the tree. The advantages for finding integer solutions early are:
  - It is better to have at least a feasible solution for the original problem (in case that we want to abort the tree building),
  - Secondly, identifying such a feasible solution may result in subsequent nodes become leaves (because of pruning by bound).
- It is very simple to code a recursive algorithm based on the *dfs* strategy.
- While building the tree, the subsequent LP problems are obtained by adding or refining a bound on one specific variable. This leads us to the observation that one can reuse the optimal simplex tableau for an LP problem for solving one of its two subsequent problems.

- The algorithm builds a tree having nodes $N_0, N_1, \ldots$
- For every node $N_i$ we associate an MILP problem $(P_i)$ and its natural linear relaxation $(R_i)$.
- To the root $N_0$ we associate the original problem $(P)$ and its relaxation $(R)$.
- $z_i$ is the optimal value of $(R_i)$ (if feasible).
- $\overline{z}$ is the best-so-far lower bound for the optimal value of $(P)$, $z_*$.
- $(\mathrm{x}^*, \mathrm{y}^*)$ is the best-so-far feasible solution for $(P)$.
- $\mathcal{L}$ is a list containing the nodes that must be still be solved. Usually $\mathcal{L}$ would be a stack (for a *dfs* strategy).

0. *Initialize.* $\mathcal{L} = \{N_0\}$, $\overline{z} = -\infty$, $(\mathrm{x}^*, \mathrm{y}^*) = \varnothing$.
1. *Terminate?* If $\mathcal{L} = \varnothing$, then $(\mathrm{x}^*, \mathrm{y}^*)$ is an optimal solution.
2. *Select node.* Choose a node $N_i$ in $\mathcal{L}$ and delete it from $\mathcal{L}$.
3. *Bound.* If $(R_i)$ is infeasible, go to step 1. Otherwise let $(\mathrm{x}^i, \mathrm{y}^i)$ be an optimal solution to $(R_i)$, and $z_i$ its optimal value.
4. *Prune.* If $z_i \leqslant \overline{z}$, go to step 1. Otherwise, if $(\mathrm{x}^i, \mathrm{y}^i)$ is feasible to $(P)$ (i. e., $\mathrm{x}^i \in \mathbb{Z}^n$), then set $\overline{z} = z_i$ and $(\mathrm{x}^*, \mathrm{y}^*) = (\mathrm{x}^i, \mathrm{y}^i)$, and go to the step 1.
5. *Branch.* Choose $x_j^i \notin \mathbb{Z}$. Create two new sub-problems $(P_{i_1})$ and $(P_{i_2})$, by branching on the variable $x_j$: one with the additional constraint $x_j \leqslant \lfloor x_j^i \rfloor$, the other with $x_j \geqslant \lceil x_j^i \rceil$. Add the corresponding nodes $N_{i_1}$ and $N_{i_2}$ to $\mathcal{L}$, and go to the step 1.

- The cutting plane method was introduced by R. Gomory (1958).
- The basic idea is to solve the natural linear LP relaxation of the (pure) ILP problem. If the resulting solution is integer (that is, all variables have integer values), then we are finished.
- Otherwise we add a constraint that cuts (eliminates) some infeasible solutions (including the one just obtained).
- The new constraint is constructed in such a way that it not eliminate any feasible integer solution of the original problem.
- The new problem is solved and the process is iterated. After adding a number of constraints, we eventually find an optimal solution to the ILP problem.

- We consider again a (pure) ILP problem, this time in standard form, and its natural linear relaxation:

$$\begin{aligned} \text{maximize} \quad & z = c^T x, \\ \text{subject to} \quad & Ax = b, \\ & x \geqslant 0 \\ & x \in \mathbb{Z}^n. \end{aligned} \tag{13}$$

$$\begin{aligned} \text{maximize} \quad & z = c^T x, \\ \text{subject to} \quad & Ax = b, \\ & x \geqslant 0. \end{aligned} \tag{14}$$

1. *Solve.* Solve the appropriate natural LP relaxation problem. Let $x^*$ be an optimal solution to this problem. If the solution, $x^*$, is an integer vector, then stop.
2. *Cutting Plane.* Otherwise generate a cutting plane constraint separating $x^*$ from the feasible region (a constraint which is satisfied by all integer solutions of the problem, but not by $x^*$).
3. *Refine the Problem.* Add the cutting plane constraint to the ILP problem and go to step 1.

- Let $\mathrm{x}^0$ be an optimal basic feasible solution to the problem (14) with optimal value $z_0$. Suppose also that the base corresponding to this solution is $B$.
- From the optimal tableau we get the equations corresponding to basic variables

$$\mathrm{x}_B + \mathrm{B}^{-1}\mathrm{N}\mathrm{x}_N = \mathrm{B}^{-1}\mathrm{b}.$$

- Define $\tilde{a}_{ij} = (\mathrm{B}^{-1}\mathrm{A}_j)_i$, for all $i \in B$ and $j \in N$, and $\tilde{b} = \mathrm{B}^{-1}\mathrm{b}$.
- If every basic variable $x_i$ has an integer value (that is, $x_i^0 \in \mathbb{Z}, \forall i \in B$), then we are finished.

# Gomory fractional cut



- Otherwise, there exist an $i \in B$ with $x_i^0 \notin \mathbb{Z}$.
- The equation labeled with $x_i$ is

$$x_i + \sum_{j \in N} \tilde{a}_{hj} x_j = \tilde{b}_h = x_i^0 \notin \mathbb{Z}.$$

- Since $x_j \geqslant 0, \forall j \in N$, and $\lfloor \tilde{a}_{hj} \rfloor \leqslant \tilde{a}_{hj}, \forall j \in N$, we must have

$$x_i + \sum_{j \in N} \lfloor \tilde{a}_{hj} \rfloor x_j \leqslant x_i + \sum_{j \in N} \tilde{a}_{hj} x_j = \tilde{b}_h$$

- For any solution $\mathrm{x} \in \mathbb{Z}^n$ of problem (13) we must have

$$x_i + \sum_{j \in N} \lfloor \tilde{a}_{hj} \rfloor x_j \leqslant \lfloor \tilde{b}_h \rfloor. \qquad (15)$$

- The inequation (15) must be satisfied by every integer solution of the problem (13).
- But $\mathrm{x}^0$ doesn't satisfy it because $x_i^0 = \tilde{b}_h$, and $x_j = 0$ for all $j \in N$, and $\tilde{b}_h$ is fractional.
- Inequation (15) is a *Gomory fractional cut*.
- It was proved (for a pure ILP) that by systematically adding these cuts, the finite convergence of the above method is ensured.

Consider the following ILP problem

$$(P)\begin{cases}\min & x_1 - 2x_2 \\ \text{s. t.} & -4x_1 + 6x_2 \leqslant 9 \\ & x_1 + x_2 \leqslant 4 \\ & x_1, x_2 \in \mathbb{Z}_+\end{cases}.$$

Its natural LP relaxation is

$$(R)\begin{cases}\min & x_1 - 2x_2 \\ \text{s. t.} & -4x_1 + 6x_2 \leqslant 9 \\ & x_1 + x_2 \leqslant 4 \\ & x_1, x_2 \geqslant 0\end{cases}.$$

Its standard form is

$$(R_1)\begin{cases} \min & x_1 - 2x_2 \\ \text{s. t.} & -4x_1 + 6x_2 + x_3 = 9 \\ & x_1 + x_2 + x_4 = 4 \\ & x_1, x_2, x_3, x_4 \geqslant 0 \end{cases}.$$

We solve $(R_1)$ using Simplex algorithm (Two Phase Method is not really necessary here). The last tableau is

Table: Last Tableau for $(R_1)$.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | RHS |
|---|---|---|---|---|---|
| $x_2$ | 0 | 1 | 1/10 | 4/10 | 25/10 |
| $x_1$ | 1 | 0 | -1/10 | 6/10 | 15/10 |
| $z$ | 0 | 0 | 3/10 | 2/10 | 35/10 |

Let us consider the first equation (corresponding to a non-integer value):

$$x_2 + \frac{1}{10}x_3 + \frac{4}{10}x_4 = \frac{25}{10}.$$

The associated Gomory fractional cut is $x_2 \leqslant 2$. We add the constraint to our LP relaxation and get

$$(R_2)\begin{cases} \min & x_1 - 2x_2 \\ \text{s. t.} & -4x_1 + 6x_2 + x_3 = 9 \\ & x_1 + x_2 + x_4 = 4 \\ & x_2 + x_5 = 2 \\ & x_1, x_2, x_3, x_4, x_5 \geqslant 0 \end{cases}.$$

We solve $(R_2)$:

Table: Last Tableau for $(R_2)$.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| $x_2$ | 0 | 1 | 0 | 0 | 1 | 2 |
| $x_4$ | 0 | 0 | 1/4 | 1 | -5/2 | 5/4 |
| $x_1$ | 1 | 0 | -1/4 | 0 | 3/2 | 3/4 |
| $z$ | 0 | 0 | 1/4 | 0 | 1/2 | 13/4 |

Let us consider the last equation (corresponding to a non-integer value):

$$x_1 - \frac{1}{4}x_3 + \frac{3}{2}x_5 = \frac{3}{4}.$$

The new Gomory cut is $x_1 - x_3 + x_5 \leqslant 0$, which, in terms of original variables becomes

$$-3x_1 + 5x_2 \leqslant 7.$$

We add the constraint to our LP relaxation $(R_2)$ and get

$$(R_3)\begin{cases} \min & x_1 - 2x_2 & \\ \text{s. t.} & -4x_1 + 6x_2 + x_3 & = 9 \\ & x_1 + x_2 + x_4 & = 4 \\ & x_2 + x_5 & = 2 \\ & -3x_1 + 5x_2 + x_6 & = 7 \\ & x_1, x_2, x_3, x_4, x_5, x_6 & \geqslant 0 \end{cases}.$$

Solve $(R_3)$ and find a feasible integer solution $x_1 = 1, x_2 = 2$, which must be an optimal feasible (integer) solution for our original problem:

Table: Last Tableau for $(R_3)$.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|
| $x_3$ | 0 | 0 | 1 | 0 | 2/3 | -4/3 | 1 |
| $x_4$ | 0 | 0 | 0 | 1 | -8/3 | 1/3 | 1 |
| $x_1$ | 1 | 0 | 0 | 0 | 5/3 | -1/3 | 1 |
| $x_2$ | 0 | 1 | 0 | 0 | 1 | 0 | 2 |
| $z$ | 0 | 0 | 0 | 0 | 1/3 | 1/3 | 3 |

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