Operations Research - Lecture 5

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Operations Research Operations Research Operations Research Operations Research

Table of contents

Operations Research

Naïve motivation

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- Consider the following canonical form LP problem

$$\begin{array}{lrcl} \text{minimize} & z = 4x_1 + x_2 + 3x_3 & & \\ \text{subject to} & x_1 - 2x_2 + x_3 & \geqslant & 2 \\ & 2x_1 + x_2 + x_3 & \geqslant & 3 \\ & x_1, x_2, x_3 & \geqslant & 0 \end{array} \tag{1}$$

- Every feasible solution gives an upper bound to the optimal objective function value: the solution $(x_1, x_2, x_3) = (2, 0, 0)$ says that $z_* \leqslant 8$.
- Now, if we multiply the first constraint by 2 and add it to the second constraint we get

$$\begin{array}{lrcl} & 2 \cdot (x_1 - 2x_2 + x_3) & \geqslant & 2 \cdot 2 \\ + & 2x_1 + x_2 + x_3 & \geqslant & 3 \\ \hline & 4x_1 - 3x_2 + 3x_3 & \geqslant & 7 \end{array}$$

Systematic Approach

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- Comparing the last expression with the objective function we get

$$4x_1 + x_2 + 3x_3 \geqslant 4x_1 - 3x_2 + 3x_3 \geqslant 7,$$

for all $\mathrm{x} \in \mathbb{R}^3_+$. Therefore, $7 \leqslant z_* \leqslant 8$.

- A more systematic motivation will lead us to multiply the constraints not by specific numbers, but by variables, say y_1 and y_2.
- After that we can try to find values for this variables that gives us the best (largest) lower bound for optimal objective function value.
- Generally, following this procedure, we get $(y_1, y_2 \geqslant 0)$:

$$\begin{array}{lrcr} & y_1 \cdot (x_1 - 2x_2 + x_3) & \geqslant & y_1 \cdot 2 \\ + & y_2 \cdot (2x_1 + x_2 + x_3) & \geqslant & y_2 \cdot 3 \\ \hline & (y_1 + 2y_2)x_1 + (-2y_1 + y_2)x_2 + (y_1 + y_2)x_3 & \geqslant & 2y_1 + 3y_2 \end{array}$$

Systematic Approach

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- Now, we compare this sum (by seeing it as a lower bound) with the objective function:

$$z = 4x_1 + x_2 + 3x_3 \geqslant (y_1 + 2y_2)x_1 + (-2y_1 + y_2)x_2 + (y_1 + y_2)x_3 \geqslant 2y_1 + 3y_2$$

- Furthermore, we impose that every coefficient of x_i to be as small as the corresponding coefficient of x_i in the objective function:

$$\begin{aligned} y_1 + 2y_2 &\leqslant 4 \\ -2y_1 + y_2 &\leqslant 1 \\ y_1 + y_2 &\leqslant 3 \end{aligned}$$

Systematic Approach

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- We found ourselves in the face of a new optimization (maximization) problem:

$$\begin{array}{lrcl} \text{maximize} & w = 2y_1 + 3y_2 & & \\ \text{subject to} & y_1 + 2y_2 & \leqslant & 4 \\ & -2y_1 + y_2 & \leqslant & 1 \\ & y_1 + y_2 & \leqslant & 3 \\ & y_1, y_2 & \geqslant & 0 \end{array} \qquad (2)$$

- This problem is called the *dual LP problem* associated with problem (1).
- The above procedure is called *Lagrange multiplier method* - a more general method used for mimimize a function under some (equation) constraints.

Definition of Dual Problem

Consider the problem in canonical form (for a minimization problem), called the primal problem:

$$\begin{array}{rl} \text{minimize} & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} \geqslant \mathbf{b} \\ & \mathbf{x} \geqslant 0. \end{array} \tag{3}$$

By definition the associated dual problem of (3) is

$$\begin{array}{rl} \text{maximize} & w = \mathbf{b}^T\mathbf{y} \\ \text{subject to} & A^T\mathbf{y} \leqslant \mathbf{c} \\ & \mathbf{y} \geqslant 0. \end{array} \tag{4}$$

This problem is in the other canonical form (for a maximization problem).

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Properties Related to Duality

Definition

A LP maximization problem is said to be in canonical form if it is presented like problem (4). That is, all the constraints are "$\leqslant$", and all variables are nonnegative.

- We know that every LP problem can be converted to a minimization problem and then converted in a canonical form.
- Hence, if we see the dual problem as a minimization problem in its canonical form, we can dualize it again. It is no surprize that, by doing so, we get the primal problem.

Lemma

The dual of the dual problem is the primal problem.

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Properties Related to Duality

Proof. First, let us convert problem (4) to a minimization problem in canonical form

$$\begin{aligned} \text{minimize} \quad & w' = -\mathrm{b}^T \mathrm{y} \\ \text{subject to} \quad & -\mathrm{A}^T \mathrm{y} \geqslant -\mathrm{c} \\ & \mathrm{y} \geqslant 0. \end{aligned}$$

The dual of this problem is

$$\begin{aligned} \text{maximize} \quad & z' = -\mathrm{c}^T \mathrm{x} \\ \text{subject to} \quad & (-\mathrm{A}^T)^T \mathrm{x} \leqslant -\mathrm{b} \\ & \mathrm{x} \geqslant 0. \end{aligned}$$

Which is equivalent to

$$\begin{aligned} \text{minimize} \quad & z = \mathrm{c}^T \mathrm{x} \\ \text{subject to} \quad & \mathrm{A}\mathrm{x} \geqslant \mathrm{b} \\ & \mathrm{x} \geqslant 0, \end{aligned}$$

that is, the primal problem (3). □

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General Rules of Duality

- As we already pointed out, every LP problem can be transformed into a minimization problem and then converted in canonical form.
- Therefore, any LP problem has a dual. The rules of general duality will be deduced after we will apply these transformations.
- Let us consider an LP problem with non-negative variables, that is

$$\begin{array}{ll} \text{minimize} & z = c^T x \\ \text{subject to} & A_1 x \geqslant b_1 \\ & A_2 x \leqslant b_2 \\ & A_3 x = b_3 \\ & x \geqslant 0 \end{array} \qquad (5)$$

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General Rules of Duality

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- We convert it to a canonical form

$$\begin{array}{lrcl} \text{minimize} & z = \mathrm{c}^T\mathrm{x} & & \\ \text{subject to} & \mathrm{A}_1\mathrm{x} & \geqslant & \mathrm{b}_1 \\ & -\mathrm{A}_2\mathrm{x} & \geqslant & -\mathrm{b}_2 \\ & \mathrm{A}_3\mathrm{x} & \geqslant & \mathrm{b}_3 \\ & -\mathrm{A}_3\mathrm{x} & \geqslant & -\mathrm{b}_3 \\ & & \mathrm{x} \geqslant 0 & \end{array}$$

- We must define four groups of dual variables: one for every group of constraints: $\mathrm{y}_1, \mathrm{y}_2', \mathrm{y}_3', \mathrm{y}_3''$. The dual problem is

$$\begin{array}{ll} \text{maximize} & w = \mathrm{b}_1^T\mathrm{y}_1 - \mathrm{b}_2^T\mathrm{y}_2' + \mathrm{b}_3^T\mathrm{y}_3' - \mathrm{b}_3^T\mathrm{y}_3'' \\ \text{subject to} & \mathrm{A}_1^T\mathrm{y}_1 - \mathrm{A}_2^T\mathrm{y}_2' + \mathrm{A}_3^T\mathrm{y}_3' - \mathrm{A}_3^T\mathrm{y}_3'' \leqslant \mathrm{c} \\ & \mathrm{y}_1, \mathrm{y}_2', \mathrm{y}_3', \mathrm{y}_3'' \geqslant 0 \end{array} \tag{6}$$

General Rules of Duality

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- If we change the notations, by putting $y_2 = -y_2'$ and $y_3 = y_3' - y_3''$, then we have an equivalent form of the dual

$$\begin{array}{ll} \text{maximize} & w = b_1^T y_1 + b_2^T y_2 + b_3^T y_3 \\ \text{subject to} & A_1^T y_1 + A_2^T y_2 + A_3^T y_3 \leqslant c \\ & y_1 \geqslant 0, y_2 \leqslant 0, y_3 \text{ unrestricted} \end{array}$$

- The directions of the constraints in the primal problem are not canonical, this implies that the signs of the variables in the dual problem are not canonical. The rules are
  - the dual variables associated with "$\geqslant$" constraints are nonnegative;
  - the dual variables associated with "$\leqslant$" constraints are nonpositive;
  - the dual variables associated with equations are unrestricted.

General Rules of Duality

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- Now, let us onsider a LP problem having only the constraints in canonical form, i. e.,

$$\begin{array}{ll} \text{minimize} & z = c_1^T x_1 + c_2^T x_2 + c_3^T x_3 \\ \text{subject to} & A_1 x_1 + A_2 x_2 + A_3 x_3 \geqslant b \\ & x_1 \geqslant 0, x_2 \leqslant 0, x_3 \text{ unrestricted} \end{array} \tag{7}$$

- We put this problem in canonical form by converting the variables, $x_2' = -x_2$ and $x_3 = x_3' - x_3''$:

$$\begin{array}{ll} \text{minimize} & z = c_1^T x_1 - c_2^T x_2' + c_3^T x_3' - c_3^T x_3' \\ \text{subject to} & A_1 x_1 - A_2 x_2' + A_3 x_3' - A_3 x_3'' \geqslant b \\ & x_1, x_2', x_3', x_3'' \geqslant 0 \end{array}$$

- The dual of the later problem is

$$\begin{array}{ll} \text{maximize} & w = b^T y \\ \text{subject to} & A_1^T y \leqslant c_1 \\ & -A_2^T y \leqslant -c_2 \\ & A_3^T y \leqslant c_3 \\ & -A_3^T y \leqslant -c_3 \\ & y \geqslant 0 \end{array}$$

- Which is equivalent with

$$\begin{array}{ll} \text{maximize} & w = b^T y \\ \text{subject to} & A_1^T y \leqslant c_1 \\ & A_2^T y \geqslant c_2 \\ & A_3^T y = c_3 \\ & y \geqslant 0 \end{array} \tag{8}$$

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General Rules of Duality

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- We see that the signs of the variables in the primal problem influence the types of constraints in the dual problem. The rules are:
  - the dual constraints associated with nonnegative variables are "$\leqslant$", i.e., consistent with canonical form;
  - the dual constraints associated with nonpositive variables are "$\geqslant$"i.e., reversed from canonical form;
  - the dual constraints associated with unrestricted variables are equations.

Table: Duality Rules

primal/dual constraint		dual/primal variable
consistent with canonical form	⇔	variable nonnegative ($\geqslant$)
reversed from canonical form	⇔	variable nonpositive ($\leqslant$)
equation	⇔	variable unrestricted

General Rules of Duality

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- Consider the primal problem

$$\begin{array}{lrcr} \text{maximize} & w = 2x_1 - 3x_2 + 3x_3 + 2x_4 & & \\ \text{subject to} & 3x_1 + x_2 - 2x_3 & \geqslant & 5 \\ & 2x_1 - x_2 + 2x_3 + 2x_4 & = & 4 \\ & x_1 + x_2 + x_3 - x_4 & \leqslant & 2 \\ & x_1 \geqslant 0, x_2, x_4 \leqslant 0, x_3 \text{ unrestricted} & & \end{array} \tag{9}$$

- Its dual is

$$\begin{array}{lrcr} \text{minimize} & z = 5y_1 + 4y_2 + 2y_3 & & \\ \text{subject to} & 3y_1 + 2y_2 + y_3 & \geqslant & 2 \\ & y_1 - y_2 + y_3 & \leqslant & -3 \\ & -2y_1 + 2y_2 + y_3 & = & 3 \\ & 2y_2 - y_3 & \leqslant & 2 \\ & y_1 \leqslant 0, y_2 \text{ unrestricted}, y_3 \geqslant 0 & & \end{array} \tag{10}$$

Weak Duality

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- Consider the following pair of primal/dual problems

$$\left\{\begin{array}{ll} \text{minimize} & z = \mathrm{c}^T\mathrm{x} \\ \text{subject to} & \mathrm{A}\mathrm{x} = \mathrm{b} \\ & \mathrm{x} \geqslant 0. \end{array}\right. \qquad \left\{\begin{array}{ll} \text{maximize} & w = \mathrm{b}^T\mathrm{y} \\ \text{subject to} & \mathrm{A}^T\mathrm{y} \leqslant \mathrm{c} \\ & \mathrm{y} \text{ unrestricted.} \end{array}\right. \tag{11}$$

Theorem

(Weak Duality) *Let* x *be a feasible solution for the primal problem and* y *be a feasible solution for the dual problem. Then*

$$z = \mathrm{c}^T\mathrm{x} \geqslant \mathrm{b}^T\mathrm{y} = w.$$

Weak Duality

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Proof. Using the inequalities from both, the primal and the dual problems, and the nonnegativity of all variables $z = \mathrm{c}^T\mathrm{x} \geqslant (\mathrm{A}^T\mathrm{y})^T\mathrm{x} = \mathrm{y}^T\mathrm{A}\mathrm{x} = \mathrm{y}^T\mathrm{b} = \mathrm{b}^T\mathrm{y} = w$. □

- The importance of the Weak Duality Theorem comes from the following consequence.

Corollary

(i) *Unboundedness of the primal problem implies the infeasibility of the dual.*

(ii) *Let* x *be a feasible solution for the primal problem and* y *be a feasible solution for the dual problem, such that* $\mathrm{c}^T\mathrm{x} = \mathrm{b}^T\mathrm{y}$*. Then* x *and* y *are optimal for their respective problems.*

Weak Duality

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Proof. (i) If the primal problem is unbounded, then it exists a sequence of feasible solutions $\left(x^k\right)_{k \geqslant 0}$, such that $\lim\limits_{k \to \infty} c^T x^k = -\infty$.

Suppose, on the contrary, that the dual problem has a feasible solution y. From Weak Duality Theorem we have $c^T x^k \geqslant b^T y$, for all $k \geqslant 0$. Hence, $-\infty \geqslant b^T y$ - which is a contradiction: the dual problem is infeasible.

(ii) If both problems are feasible, then they are both bounded; let x_* and y_* two optimal solutions for their respective problems, primal and dual. From the following sequence or relations

$$c^T x \geqslant c^T x_* \geqslant b^T y_* \geqslant b^T y = c^T x,$$

follows that $c^T x = c^T x_* = b^T y_* = b^T y$. □

Weak Duality - Example

- Consider the following pair of primal/dual problems

$$(P)\begin{cases} \max & z = 2x_1 + 3x_2 \\ \text{s. t.} & x_1 - 2x_2 \leqslant 2 \\ & x_1 + x_2 \geqslant 1 \\ & x_1, x_2 \geqslant 0 \end{cases} \quad (D)\begin{cases} \min & w = 2y_1 + y_2 \\ \text{s. t.} & y_1 + y_2 \geqslant 2 \\ & -2y_1 + y_2 \geqslant 3 \\ & y_1 \geqslant 0, y_2 \leqslant 0 \end{cases}$$

  Since (P) is unbounded, (D) must be infeasible (*check yourself!*).

- Now, consider another pair of primal/dual problems

$$(P')\begin{cases} \min & z = -x_1 + 2x_2 \\ \text{s. t.} & x_1 - x_2 \geqslant 2 \\ & 2x_1 - 2x_2 \leqslant 1 \\ & x_1, x_2 \geqslant 0 \end{cases} \quad (D')\begin{cases} \max & w = 2y_1 + y_2 \\ \text{s. t.} & y_1 + 2y_2 \leqslant -1 \\ & -y_1 - 2y_2 \leqslant 2 \\ & y_1 \geqslant 0, y_2 \leqslant 0 \end{cases}$$

  Here (P) is infeasible, hence (D) can be infeasible or (feasible and) unbounded (see the Strong Duality Theorem; *test the two possibilities!*).

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Weak Duality

- If x and y are feasible solutions for primal and dual problems, respectively, the *duality gap between x and y* is the difference between the objective functions values corresponding to these solutions:

$$c^T x - b^T y \geqslant 0.$$

- The *duality gap between the two problems* is the duality gap between two optimal solutions for the pair of primal/dual problems provided that the two problems are bounded.
- Corollary 3.1 says that if the duality gap between two solutions is zero, then the two solutions are optimal for their respective problems.
- The next result will show that in most of the cases the duality gap between problems is zero, because this is equivalent to the fact that one of the problems has a (finite) optimal feasible solution.

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Strong Duality

Theorem

(Strong Duality) *If the primal problem or the dual problem has an optimal solution then so does the other, and the optimal objective values are equal.*

Proof. Without restrain the generality we can assume that the primal problem has an optimal solution and that this problem is in standard form.

Let x_* be an optimal basic feasible solution for the primal (obtained using the simplex algorithm). By reordering the variables we can suppose that $x_*^T = (x_B^T \ x_N^T)$. In line with this separation, we write $A = (B \ N)$ and $c^T = (c_B^T \ c_N^T)$.

Strong Duality

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Then, $x_B = B^{-1}b$ and, since x_* is optimal, the reduced costs of nonbasic variables are nonnegative: $c_N^T - c_B^T B^{-1} N \geqslant 0$.

Let $y_* = (B^{-1})^T c_B$. y_* is feasible solution for the dual:

$$A^T y_* = \begin{pmatrix} B^T \\ N^T \end{pmatrix} \cdot y_* = \begin{pmatrix} B^T y_* \\ N^T y_* \end{pmatrix} = \begin{pmatrix} B^T (B^{-1})^T c_B \\ N^T (B^{-1})^T c_B \end{pmatrix} =$$

$$\begin{pmatrix} (B^{-1}B)^T c_B \\ N^T (B^{-1})^T c_B \end{pmatrix} = \begin{pmatrix} c_B \\ (c_B^T B^{-1} N)^T \end{pmatrix} \leqslant \begin{pmatrix} c_B \\ c_N \end{pmatrix} = c.$$

On the other hand

$$z = c^T x_* = c_B^T x_B = c_B^T B^{-1} b = y_*^T b = b^T y_* = w.$$

Using corollary (3.1) (ii) we get the desired conclusion. □

Complementary Slackness

- The Strong Duality Theorem holds for every pair of primal/dual problems (given in any forms). This is true because any problem can be equivalently converted to one in standard form.
- There is a deeper relation between the nonnegativity variables in the primal problem and the constraints in the dual problem.
- This relation is called *complementary slackness* and says that its impossible to have both $x_i > 0$ and $(\mathrm{A}^T\mathrm{y})_i < c_i$, where x and y are optimal solutions for their respective problems.
- This property may help to recover an optimal solution for the dual problem from an optimal solution for the primal problem (and viceversa).

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Complementary Slackness

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Theorem

(Complementary Slackness) If x *is a feasible solution for the primal problem and* y *is a feasible solution for the dual problem, then the following are equivalent*

(i) $\mathrm{x}^T(\mathrm{c} - \mathrm{A}^T\mathrm{y}) = 0$.

(ii) x *and* y *are optimal solutions for their respective problems.*

Proof. (i) $\Rightarrow$ (ii) If $\mathrm{x}^T(\mathrm{c}-\mathrm{A}^T\mathrm{y}) = 0$, then $\mathrm{c}^T\mathrm{x} = \mathrm{b}^T\mathrm{y}$ and the conclusion follows from Corrolary (3.1).

(ii) $\Rightarrow$ (i). We know that $z = \mathrm{c}^T\mathrm{x} \geqslant \mathrm{y}^T\mathrm{A}\mathrm{x} = \mathrm{y}^T\mathrm{b} = \mathrm{b}^T\mathrm{y}$; from the Strong Duality Theorem, $\mathrm{c}^T\mathrm{x} = \mathrm{b}^T\mathrm{y}$, therefore, $\mathrm{c}^T\mathrm{x} = \mathrm{y}^T\mathrm{A}\mathrm{x}$ which yields $\mathrm{x}^T(\mathrm{c} - \mathrm{A}^T\mathrm{y}) = 0$. □

Complementary Slackness - Example

- Consider the following primal LP problem and its dual

$$(P)\begin{cases} \min \quad z = 13x_1 + 10x_2 + 6x_3 \\ \text{s. t.} \quad 5x_1 + x_2 + 3x_3 = 8 \\ \qquad 3x_1 + x_2 = 3 \\ \qquad x_1, x_2, x_3 \geqslant 0 \end{cases} \quad (D)\begin{cases} \max \quad w = 8y_1 + 3y_2 \\ \text{s. t.} \quad 5y_1 + 3y_2 \leqslant 13 \\ \qquad y_1 + y_2 \leqslant 10 \\ \qquad 3y_1 \leqslant 6 \end{cases}$$

- We will verify if $\mathrm{x} = (1\ 0\ 1)^T$ is an optimal feasible solution for the primal problem.
- Assume that, indeed, x is optimal; as $x_1, x_3 > 0$, we must have $5y_1 + 3y_2 = 13$ and $3y_1 = 6$ which yields $y_1 = 2, y_2 = 1$.
- Now, we compute the objective functions values: $\mathrm{c}^T\mathrm{x} = 19 = \mathrm{b}^T\mathrm{y}$ - this equality says that x is optimal for the primal problem (and y is optimal for the dual problem).

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Duality Intepreted

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- The dual problem can be used to improve the interpetation of the model for the original problem at hand.
- This interpretation may vary from problem to problem; we will review this approach by using an example.
- Example: A bakery makes and sells two types of cakes, one simple and a more fancy one. These products require basic ingredients (flour, sugar, eggs etc), and some decorations and flavors (fruits, nuts etc) with the fancier cake using more of the decorations, but also more labor force. The baker would like to maximize profit.
- An LP problem associated to this situation follows.

Duality Intepreted

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$$(P)\begin{cases} \max & z = 24x_1 + 14x_2 \\ \text{s. t.} & 3x_1 + 2x_2 \leqslant 120 \\ & 4x_1 + x_2 \leqslant 100 \\ & 2x_1 + x_2 \leqslant 70 \\ & x_1, x_2 \geqslant 0 \end{cases}$$

- x_1 and x_2 represent the number of batches of the fancier and simple cakes produced per day.
- The first constraint is associated with the daily limits on the avalaibility of basic ingredients (a batch of simple cake requires 2 pounds, a batch of fancier cake requires 3 pounds).
- In a similar manner, the second constraint represents the limits on decorations and the third constraint records the work force avalaibility (1 hour/batch of simple cake vs. 2 hours/batch of fancier cake).

Duality Intepreted

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$$(D)\begin{cases} \min & w = 120y_1 + 100y_2 + 70y_3 \\ \text{s. t.} & 3y_1 + 4y_2 + 2y_3 \quad \geqslant \quad 24 \\ & 2y_1 + y_2 + y_3 \quad \geqslant \quad 14 \\ & y_1, y_2, y_3 \geqslant 0 \end{cases}$$

- The optimal feasible solution for (P) is $\mathrm{x}_* = (16\ 36)^T$, with $z_* = 888$. The optimal feasible solution for (D) is $\mathrm{y}_* = (6.4\ \ 1.2\ \ 0)^T$, with $w_* = 888$. (Complementary slackness conditions are satisfied and optimal objective values are equal.)
- The limiting factors in this problem are the resources (basic ingredients, decorations, labor force). The bakery might want to hire some new employers or to buy additional quantities of ingredients.
- In this case how much the bakery should pay?

Duality Intepreted

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- Each extra pound of ingredients will be worth $y_1 = 6.4\$$ in profit and each extra pound of decorations will be worth $y_2 = 1.2\$$ in profit.
- Additional work force is of no value, since there the labor force is in excess. This argument fails if the bakery produces too much fancy cake batches because it can drain up the labor force.

Duality Intepreted

- Another interpretation of the dual problem: if someone wants to takeover the bakery, what price should be offered?
- First the potential buyer records the values of the bakery's assets: ingredients (y_1), decorations (y_2), workforce (y_3).
- The buyer wants to minimize this total value

$$\text{minimize } w = 120y_1 + 100y_2 + 70y_3$$

- Such a price would be fair if the bakery gives a profit greater or equal to the profit obtained by producing cakes

$$\begin{aligned} 3y_1 + 4y_2 + 2y_3 &\geqslant 24 \\ 2y_1 + y_2 + y_3 &\geqslant 14 \end{aligned}$$

- The dual problem helps us to determine the daily value of the bakery's bussiness.

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Dual Simplex Algorithm

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- The Simplex method we already studied will be referred as primal simplex algorithm.
- This algorithm starts with a basic feasible solution to the primal problem and iterates until the *primal optimality conditions* are fulfilled.
- It is possible to apply the simplex algorithm to the dual problem, starting with a feasible solution to the dual problem and iterating until the *dual optimality conditions* are satisfied.

Dual Simplex Algorithm

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- The proof of the Strong Duality Theorem 3.2 shows that primal optimality conditions for the primal problem ($c_N^T - c_B^T B^{-1} N \geqslant 0$) are equivalent with the *dual feasibility conditions* ($A^T y \leqslant c$).
- The primal simplex algorithm goes through a sequence of primal feasible but dual infeasible bases, trying at each iteration to "reduce" the dual infeasibility, until the dual feasibility conditions are satisfied.
- The dual simplex algorithm will work in a dual fashion: it goes through a sequence of dual feasible but primal infeasible bases, trying at each iteration to "reduce" the primal infeasibility, until the primal feasibility conditions are satisfied.
- When these conditions are satisfied the duality theorems ensure us that an optimal dual basis was reached.

Algebra of Dual Simplex Algorithm

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- Let us suppose that we have an initial dual basic feasible solution $\mathrm{x}_B = \mathrm{B}^{-1}\mathrm{b}$ (that is, all reduced costs are nonnegative: $\widehat{c}_j \geqslant 0$, for all $j \in N$);
- For the primal point of view such a solution is *super-optimal*, since it gives an objective function value greater than optimum, without being necessarily primal feasible.
- The dual algorithm ends when the current basis becomes dual optimal, or, equivalently, primal feasible.
- In a common iteration, suppose that the current basis is not primal feasible, this corresponds to a negative right-hand side entry $\widehat{b}_k$; the k constraint is

$$(\mathrm{x}_B)_k + \sum_{i \in N} \widehat{a}_{ki} x_i = \widehat{b}_k < 0.$$

Algebra of Dual Simplex Algorithm

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- If a non-basic variable x_i $(i \in N)$ were to replace $(\mathrm{x}_B)_k$ in the new basis, then the value for x_i will be $\widehat{b}_k / \widehat{a}_{ki} > 0$ - which makes sense as long as, in this way, we reduce the infeasibility.
- The new reduced costs will be

$$\widehat{\widehat{c}}_h = \widehat{c}_h - \widehat{c}_i \frac{\widehat{a}_{kh}}{\widehat{a}_{ki}}, \forall h = \overline{1, n},$$

since we must preserve the primal optimality, we impose

$$\widehat{\widehat{c}}_h \geqslant 0 \Leftrightarrow \frac{\widehat{c}_h}{\widehat{a}_{kh}} \leqslant \frac{\widehat{c}_i}{\widehat{a}_{ki}} \, (\text{for } \widehat{a}_{kh} < 0, \text{ otherwise } \widehat{\widehat{c}}_h \geqslant 0)$$

- Therefore, if x_i is the entering variable, then it must have the smallest ratio $\left|\frac{\widehat{c}_i}{\widehat{a}_{ki}}\right|$ with $\widehat{a}_{ki} < 0$.

Dual Simplex Algorithm

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The algorithm starts with with a basis matrix B, corresponding to the dual basic feasible solution, that is, $\widehat{c}_j \geqslant 0$. The algorithm follows:

The Feasibility Test. Compute $\mathrm{x}_B = \widehat{\mathrm{b}} = \mathrm{B}^{-1}\mathrm{b}$, if $\mathrm{x}_B \geqslant 0$, then the current basis is a dual optimal feasible solution. Otherwise choose $(\mathrm{x}_B)_k$, as the leaving variable, such that $\widehat{b}_k < 0$. The kth row is the pivot row.

The Main Step. Compute $\widehat{\mathrm{A}}_j = \mathrm{B}^{-1}\mathrm{A}_j$. Find an $i \in N$ such that

$$\left|\frac{\widehat{c}_i}{\widehat{a}_{ki}}\right| = \min\left\{\left|\frac{\widehat{c}_h}{\widehat{a}_{kh}}\right| : \widehat{a}_{kh} < 0, h \in N\right\}.$$

x_i will be the entering variable and $\widehat{a}_{ij}$ the *pivot* entry. If $\widehat{a}_{hj} \geqslant 0$, for all $h \in N$, then Stop - the problem has infinite optimum.

The Update Compute the new basis matrix B, the new vector of basic variables x_B, and the new reduced costs $\widehat{c}$.

Dual Simplex Algorithm - Example

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- Consider the following LP problem

$$\begin{array}{ll} \text{minimize} & z = 2x_1 + 3x_2 \\ \text{subject to} & -3x_1 + 2x_2 \leqslant -4 \\ & -x_1 - 2x_2 \leqslant -3 \\ & x_1, x_2 \geqslant 0 \end{array}$$

- In standard form the problem becomes

$$\begin{array}{ll} \text{minimize} & z = 2x_1 + 3x_2 \\ \text{subject to} & -3x_1 + 2x_2 + x_3 = -4 \\ & -x_1 - 2x_2 + x_4 = -3 \\ & x_1, x_2, x_3, x_4 \geqslant 0 \end{array}$$

Dual Simplex Algorithm - Example

- The basis $\{x_3, x_4\}$ is infeasible but the primal optimality conditions are satisfied (this is a super-optimal solution)

Table: First Dual Simplex tableau (note that $B = I_2$).

	$\underline{x_1}$	x_2	x_3	x_4	RHS
$\underline{x_3}$	$\boxed{-3}$	2	1	0	$\underline{-4}$
x_4	-1	-2	0	1	-3
z	2	3	0	0	0

$$\underset{\text{min}}{\underline{\left|\frac{2}{-3}\right|}}$$

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Dual Simplex Algorithm - Example

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Table: Second Dual Simplex tableau.

	x_1	$\underline{x_2}$	x_3	x_4	RHS
x_1	1	-2/3	-1/3	0	4/3
$\underline{x_4}$	0	$\boxed{-8/3}$	-1/3	1	$\underline{-5/3}$
z	$\underline{0}$	13/3	2/3	0	-8/3
		$\underset{\text{min}}{\underline{\left\lvert\frac{13/3}{-8/3}\right\rvert}}$	$\left\lvert\frac{2/3}{-1/3}\right\rvert$		

Table: Third Dual Simplex tableau.

	x_1	x_2	x_3	x_4	RHS
x_1	1	0	-1/4	-1/4	7/4
x_2	0	1	1/8	-3/8	5/8
z	0	0	1/8	13/8	-43/8

Dual Simplex Algorithm - Example

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- The current basis is primal feasible, hence is optimal. The dual optimal feasible solution is $y = (1/8\ 13/8)^T$.

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