Operations Research - Lecture 4

Olariu E. Florentin

October 20, 2025

Table of contents

- In the last lecture we reviewed some of the issues related with the Simplex method. Part of them are globally linked to the framework of solving an LP problem, but some of them are strictly related to the algorithm.
- The special situations we will discuss here are:
  - Degeneracy.
  - Unboundedness.
  - Multiple optimal solutions.
  - Cycling.
  - Initial basic feasible solution.

Throught this section we will consider a LP problem in standard form

$$\begin{aligned} \text{minimize} \quad & z = \mathrm{c}^T \mathrm{x}, \\ \text{subject to} \quad & \mathrm{A}\mathrm{x} = \mathrm{b}, \\ & \mathrm{x} \geqslant 0. \end{aligned} \tag{1}$$

Definition

Let x be a basic feasible solution to problem (1). x is said to be degenerate if $x_i = \widehat{b}_i = 0$*, for some* $i \in B$.

- That is *degeneracy* occurs when the current basic feasible solution has a basic variable having zero value.

- If the current basis is degenerate, it is possible that a zero value basic variable to be chosen to leave the basis.
- Degeneracy is a sign of redundancy in information; a side effect is that the value of the objective function doesn't change, hence the algorithm doesn't progress.
- If this issue occurs we may find ourselves in a more difficult situation: cycling, which is enabled by the existence of degenerate bases.

Table: An Example of Degeneracy.

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_4	0	1.5	1	1	-0.5	0	10
x_1	1	0.5	1	0	0.5	0	10
x_6	0	1	-1	0	-1	1	0
z	0	-7	-2	0	5	0	100

Definition

Problem (1) is said to be unbounded if doesn't have a finite optimal feasible solution.

- Unboundedness means that the "optimal" value of objective is $-\infty$.
- In Simplex this situation is revealed when we can't find a leaving variable.

Table: An Example of Unboundedness.

	x_1	x_2	x_3	x_4	x_5	RHS
x_1	1	0	-2	3	0	1
x_5	0	0	0	-1	1	5
x_2	0	1	-1	0	0	3
z	0	2	-2	0	0	12

Multiple (Alternative) Optimal Solutions

Definition

Problem (1) has *multiple optimal solutions* *if there exist* $\mathrm{x}^1 \neq \mathrm{x}^2$*, both optimal feasible solutions of it.*

- In Simplex framework we can detect an alternative basic optimal solution: when we have an optimal basic feasible solution with a non-basic solution having a zero reduced cost which can be introduced in the current basis.
- In this situation the non-basic variable can be introduced in the current basis and the next basis will be optimal too.
- If a non-basic variable x_j has a null reduced cost, but $a_{ij} \leqslant 0$ for all $1 \leqslant i \leqslant m$, then a different (non-basic) optimal solution can be obtained like this ($x_{i(h)}$ labels the row h):

$$x_j = \alpha > 0, x_{j'} = 0, \forall j' \in N \setminus \{j\}, \text{ and } x_{i(h)} = b_h - a_{hj}\alpha, \forall i(h) \in B.$$

Table: Simplex Example of Alternate Optimal Solutions.

	x_1	x_2	x_3	x_4	x_5	RHS		
x_1	1	3	0	3	0	1	1/3	← min
x_5	0	1	0	-1	1	5		
x_3	0	2	1	0	0	3		
z	0	2	0	0	0	12		

	x_1	x_2	x_3	x_4	x_5	RHS
x_4	1/3	1	0	1	0	1/3
x_5	1/3	2	0	0	1	16/3
x_3	0	2	1	0	0	3
z	0	2	0	0	0	12

$(1\ 0\ 3\ 0\ 5)^T$ and $(0\ 0\ 3\ 1\ 16/3)^T$ are both optimal basic feasible solutions.

Definition

A cycle occurs in the execution of the Simplex algorithm if, after a finite number of iterations, we meet an already computed tableau.

- Consider the following LP problem

$$\text{minimize} \quad z = -3/4x_1 + 20x_2 - 1/2x_3 + 6x_4 + 3$$

subject to

$$\begin{aligned} 1/4x_1 - 8x_2 - x_3 + 9x_4 &\leqslant 0 \\ 1/2x_1 - 12x_2 - 1/2x_3 + 3x_4 &\leqslant 0 \\ x_3 &\leqslant 1 \\ x_1, x_2, \ldots, x_4 &\geqslant 0 \end{aligned}$$

- In standard form, the problem becomes

$$\begin{array}{ll} \text{minimize} & z = -3/4x_1 + 20x_2 - 1/2x_3 + 6x_4 + 3 \\ \text{subject to} & \\ & 1/4x_1 - 8x_2 - x_3 + 9x_4 + x_5 = 0 \\ & 1/2x_1 - 12x_2 - 1/2x_3 + 3x_4 + x_6 = 0 \\ & x_3 + x_7 = 1 \\ & x_1, x_2, \ldots, x_7 \geqslant 0 \end{array}$$

- We will use the following rules for finding a pivot:
  - the entering variable will be that with the most negative reduced cost (sometimes called the Dantzig rule);
  - the leaving variable will be that with the smallest index among those that are eligible (for leaving).

Table: First Simplex Tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS		
x_5	1/4	-8	-1	9	1	0	0	0	0/0.25	← min
x_6	1/2	-12	-1/2	3	0	1	0	0	0/0.5	← min
x_7	0	0	1	0	0	0	0	1		
z	$-3/4$	20	-1/2	6	0	0	0	3		

Table: Second Simplex Tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS		
x_1	1	-32	-4	36	4	0	0	0		
x_6	0	4	3/2	-15	-2	1	0	0	0/4	← min
x_7	0	0	1	0	0	0	0	1		
z	0	-4	$-7/2$	33	3	0	0	3		

Table: Third Simplex Tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS		
x_1	1	0	**8**	-84	-12	8	0	0	0/8	← min
x_2	0	1	3/8	-30/8	-1/2	1/4	0	0	0/0.375	← min
x_7	0	0	1	0	0	0	0	1	1/1	
z	0	0	-2	18	1	1	0	3		

Table: Fourth Simplex Tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS		
x_3	1/8	0	1	-21/2	-3/2	1	0	0		
x_2	-3/64	1	0	**3/16**	1/16	-1/8	0	0	0/0.1875	← min
x_7	-1/8	0	0	21/2	3/2	-1	1	1	2/21	
z	1/4	0	0	-3	-2	3	0	3		

Table: Fifth Simplex Tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS		
x_3	-5/2	56	1	0	2	-6	0	0	0/2	← min
x_4	-1/4	16/3	0	1	1/3	-2/3	0	0	0/0.33	← min
x_7	5/2	-56	0	0	-2	6	1	1		
z	-1/2	16	0	0	-1	1	0	3		

Table: Sixth Simplex Tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS		
x_5	-5/4	28	1/2	0	1	-3	0	0		
x_4	1/6	-4	-1/6	1	0	1/3	0	0	0/0.33	← min
x_7	0	0	1	0	0	0	1	1		
z	-7/4	44	1/2	0	0	-2	0	3		

Table: Seventh Simplex Tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_5	1/4	-8	-1	9	1	0	0	0
x_6	1/2	-12	-1/2	3	0	1	0	0
x_7	0	0	1	0	0	0	0	1
z	-3/4	20	-1/2	6	0	0	0	3

- After six pivots we got again the initial basic solution, with the same tableau, and same basis.
- This sequence of pivots can be repeated over and over, and the algorithm never ends.

Anticycling Rules

- Obviously, this situation is induced by the degeneracy - all the intermediate bases are (and must be) degenerate (why?).
- Although the degeneracy doesn't always imply cycling, without degeneracy we cannot have cycles.
- The solution to this issue stands in choosing a certain pivoting rule, degeneracy being sometimes unavoidable - as in our example.
- We will describe below two anticycling rules: lexicographic and Bland's rule.

Definition

Let $\mathrm{u} \neq \mathrm{v} \in \mathbb{R}^n$; u *is lexicographically larger than* v, *and write* $\mathrm{u} >_L \mathrm{v}$, *if the first non-zero component of* $\mathrm{u} - \mathrm{v}$ *is positive.*

- *Lexicographic Pivoting Rule*:
  - Choose an entering variable x_j as long as its reduced cost is negative; let u be the column corresponding to x_j (i.e., the jth column).
  - For each $u_i > 0$, divide the ith row of the table by u_i, and choose the lexicographically smalest row - this will be the row of the leaving variable.
- *Bland's Rule* (smallest index pivoting rule):
  - Find the smallest index j such that the reduced cost $\widehat{c}_j$ is negative.
  - Among all the indexes k for which $\dfrac{\widehat{b}_k}{\widehat{a}_{kl}} = \min\left\{\dfrac{\widehat{b}_h}{\widehat{a}_{hl}} : \widehat{a}_{hl} > 0\right\}$, choose the minimum one - the variable which labels the kth row will be the leaving variable.

- Simplex algorithm iterates from one basic feasible solution to another until an optimal solution is found or until unboundedness is proved.
- In our examples the initial basic feasible solution is the set formed with all slack variables. This was possible because the original problem has all constraints of the form $Ax \leqslant b$ and $b \geqslant 0$.
- By introducing slack variables the constraints become $Ax + s = b$. The vector (x, s) with $s = b$ and $x = 0$ is a basic feasible solution (with $B = I$).

- Usually, problems in standard form may have constraints which doesn't contain any slack variable. In this way occurs the following question: *how to choose an initial basic feasible solution for a problem in general form?*
- This section give two answers to the above question: the Two Phase Method and the Big M Method.
- Both these methods rely on solving an auxiliary LP problem; after that we can know if the original problem has or has not an initial basic feasible solution
- That is, our methods will tell us if the original problem has or has not feasible solutions at all, since having a feasible solution means having a basic feasible solution also.

- Consider a problem in standard form ($b \geqslant 0$)

$$\begin{aligned} \text{minimize} \quad & z = c^T x, \\ \text{subject to} \quad & Ax = b, \\ & x \geqslant 0. \end{aligned} \tag{2}$$

- We introduce a vector of artificial variables $y \in \mathbb{R}^m$, that will play the role of slack variables vector, and replace the constraints with

$$\begin{aligned} & Ax + y = b, \\ & x, y \geqslant 0. \end{aligned} \tag{3}$$

- Obviously, this will be a distinct problem, and the objective function will be modified in different ways by the above methods.
- Sometimes it is not necessary to add m artificial variables, since some of the original variables can play the roles of slack variables.

- Let's use the following example

$$
\begin{aligned}
\text{minimize} \quad & z = 2x_1 + 3x_2 \\
\text{subject to} \quad & \\
& 3x_1 + 2x_2 = 14 \\
& 2x_1 - 4x_2 \geqslant 2 \\
& 4x_1 + 3x_2 \leqslant 19 \\
& x_1, x_2 \geqslant 0
\end{aligned}
$$

- In standard form the problem becomes

$$
\begin{aligned}
\text{minimize} \quad & z = 2x_1 + 3x_2 \\
\text{subject to} \quad & \\
& 3x_1 + 2x_2 = 14 \\
& 2x_1 - 4x_2 - x_3 = 2 \\
& 4x_1 + 3x_2 + x_4 = 19 \\
& x_1, x_2, x_3, x_4 \geqslant 0
\end{aligned}
$$

- We add artificial variables

$$\begin{array}{ll} \text{minimize} & z = 2x_1 + 3x_2 \\ \text{subject to} & \\ & \begin{aligned} 3x_1 + 2x_2 + y_1 &= 14 \\ 2x_1 - 4x_2 - x_3 + y_2 &= 2 \\ 4x_1 + 3x_2 + x_4 &= 19 \\ x_1, x_2, x_3, x_4, y_1, y_2 &\geqslant 0 \end{aligned} \end{array}$$

- Now, we can start the Simplex with the initial base $\{y_1, y_2, x_4\}$; note that x_4 can play the role of a slack variable, hence, two artificial variables are enough.
- But this basis doesn't correspond to a basic feasible solution of the original problem, since the artificial variables doesn't belong to the original problem.

- In the Two Phase Method, the artificial variables are used to create an auxiliary LP problem - the *phase I problem*.
- This new problem aims only to find a basic feasible solution to the original problem.
- The objective for the phase I problem is

$$\text{minimize} \quad z' = \sum_j y_j.$$

- The phase I problem is

$$\begin{aligned} \text{minimize} \quad & z' = \textstyle\sum_j y_j, \\ \text{subject to} \quad & \mathrm{A}\mathrm{x} + \mathrm{y} = \mathrm{b}, \\ & \mathrm{x}, \mathrm{y} \geqslant 0. \end{aligned} \tag{4}$$

- Let z'_* be the optimal value of the objective function for the phase I problem; note that this problem has a finite optimum, since it cannot be unbounded.
- If the original problem is feasible, then $z'_* = 0$, otherwise $z'_* > 0$. Hence, the original problem is feasible if and only if $z'_* = 0$.
- The phase I problem for our example is

$$\begin{array}{ll} \text{minimize} & z = y_1 + y_2 \\ \text{subject to} & \\ & \begin{aligned} 3x_1 + 2x_2 + y_1 &= 14 \\ 2x_1 - 4x_2 - x_3 + y_2 &= 2 \\ 4x_1 + 3x_2 + x_4 &= 19 \\ x_1, x_2, x_3, x_4, y_1, y_2 &\geqslant 0 \end{aligned} \end{array}$$

	x_1	x_2	x_3	x_4	y_1	y_2	
y_1	3	2	0	0	1	0	14
y_2	2	-4	-1	0	0	1	2
x_4	4	3	0	1	0	0	19
z'	0	0	0	0	1	1	0

- Obviously this tableau is not in a proper simplex form: we must express z' only in terms of non-basic variables, by eliminating basic (i.e., artificial) variables from their constraints:

$$y_1 = 14 - 3x_1 - 2x_2, y_2 = 2 - 2x_1 + 4x_2 + x_3,$$

$$z' = y_1 + y_2 = -5x_1 + 2x_2 + x_3 + 16.$$

Table: First Simplex Tableau - Phase I.

	x_1	x_2	x_3	x_4	y_1	y_2	RHS		
y_1	3	2	0	0	1	0	14	14/3	
y_2	2	-4	-1	0	0	1	2	2/2	← min
x_4	4	3	0	1	0	0	19	19/4	
z'	-5	2	1	0	0	0	-16		

Table: Second Simplex Tableau - Phase I.

	x_1	x_2	x_3	x_4	y_1	y_2	RHS		
y_1	0	8	3/2	0	1	-3/2	11	11/8	
x_1	1	-2	-1/2	0	0	1/2	1		
x_4	0	11	2	1	0	-2	15	15/11	← min
z'	0	-8	-3/2	0	0	5/2	-11		

Table: Third Simplex Tableau - Phase I.

	x_1	x_2	x_3	x_4	y_1	y_2	RHS		
y_1	0	0	1/22	-8/11	1	-1/22	1/11	2	← min
x_1	1	0	-3/22	2/11	0	3/22	41/11		
x_2	0	1	2/11	1/11	0	-2/11	15/11	15/2	
z'	0	0	$-1/22$	8/11	0	23/22	-1/11		

Table: Fourth Simplex Tableau - Phase I.

	x_1	x_2	x_3	x_4	y_1	y_2	RHS
x_3	0	0	1	-16	22	-1	2
x_1	1	0	0	-2	3	0	4
x_2	0	1	0	3	-4	0	1
z'	0	0	0	0	1	1	0

- After three iterations, the curent basis doesn't contain any artificial variable and the objective value is zero, hence we have a basic feasible solution for the original problem.
- We can remove the columns corresponding to artificial variables and restate the original objective function:

	x_1	x_2	x_3	x_4	RHS
x_3	0	0	1	-16	2
x_1	1	0	0	-2	4
x_2	0	1	0	3	1
z	2	3	0	0	0

- Obviously, this is not a proper form Simplex tableau, since there are some non-zero reduced costs of basic variables; we must replace these variables from their equations:

$$x_1 = 2x_4 + 4, x_2 = -3x_4 + 1,$$

$$z = 2x_1 + 3x_2 = -5x_4 + 11.$$

Table: First Simplex Tableau - Phase II.

	x_1	x_2	x_3	x_4	RHS
x_3	0	0	1	-16	2
x_1	1	0	0	-2	4
x_2	0	1	0	3	1
z	0	0	0	-5	-11

- From this point we can use Simplex to solve the original problem - this is *phase II* (left as an exercise).

- After solving Phase I problem, it may happen that the optimal value is zero, but some artificial variables are basic ones, in this case we proceed like this:
  - Let the ith basic variable (from the optimal basic feasible solution) be an artificial one, x_h.
  - We choose an $\widehat{a}_{ij} \neq 0$, where x_j is a non-basic variable from the original problem and pivot such that x_h leaves and x_j enters the basis.
  - If we can't find such a variable x_j, then we can remove the ith line (it is not relevant for the original problem) and the hth column.
  - Repeat this steps until there are no more artificial basic variables.
  - After all that, transform the Simplex tableau to proper form and apply the second phase.

- Consider the problem

$$\begin{aligned} \text{minimize} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & \\ & 2x_1 + x_2 + x_3 = 4 \\ & x_1 + x_2 + 2x_3 = 2 \\ & x_1, x_2, x_3 \geqslant 0 \end{aligned}$$

- We add artificial variables and modify the objective function

$$\begin{aligned} \text{minimize} \quad & z = y_1 + y_2 \\ \text{subject to} \quad & \\ & x_1 + x_2 + 2x_3 + y_1 = 2 \\ & 2x_1 + x_2 + x_3 + y_2 = 4 \\ & x_1, x_2, x_3, y_1, y_2 \geqslant 0 \end{aligned}$$

Table: First Simplex Tableau - Phase I.

	x_1	x_2	x_3	y_1	y_2	RHS		
y_1	1	1	2	1	0	2	2/1	$\leftarrow$ min
y_2	2	1	1	0	1	4	4/2	
z'	-3	-2	-3	0	0	-6		

Table: Second Simplex Tableau - Phase I.

	x_1	x_2	x_3	y_1	y_2	RHS
x_1	1	1	2	1	1	2
y_2	0	-1	-3	-2	1	0
z'	0	1	3	3	0	0

- The second tableau is already optimal, but the artificial y_2 remains in the basis; we eliminate it and introduce the (original) non-basic variable x_2 (the pivot is $-1 \neq 0$).

Table: Third Simplex Tableau - Phase I.

	x_1	x_2	x_3	y_1	y_2	RHS
x_1	1	0	-1	-2	2	2
x_2	0	1	3	2	-1	0
z'	0	0	0	1	1	0

- We remove the artificial variables and restate the original objective function in terms of the nonbasic variables:

Table: First Simplex Tableau - Phase II.

	x_1	x_2	x_3	RHS
x_1	1	0	-1	2
x_2	0	1	3	0
z'	0	0	-2	-2

- Now, one can proceed with the phase II (left as an exercise).

- We consider another example:

$$\begin{array}{ll} \text{minimize} & z = x_1 + 2x_2 \\ \text{subject to} & \\ & \begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 + 2x_2 &= 4 \\ x_1, x_2 &\geqslant 0 \end{aligned} \end{array}$$

- We add artificial variables and modify the objective function

$$\begin{array}{ll} \text{minimize} & z = y_1 + y_2 \\ \text{subject to} & \\ & \begin{aligned} x_1 + x_2 + y_1 &= 2 \\ 2x_1 + 2x_2 + y_2 &= 4 \\ x_1, x_2, y_1, y_2 &\geqslant 0 \end{aligned} \end{array}$$

Table: First Simplex Tableau - Phase I.

	x_1	x_2	y_1	y_2	RHS		
y_1	1	1	1	0	2	2/1	← min
y_2	2	2	0	1	4	4/2	
z'	-3	-3	0	0	-6		

Table: Second Simplex Tableau - Phase I.

	x_1	x_2	y_1	y_2	RHS
x_1	1	1	1	0	2
y_2	0	0	-2	1	0
z'	0	0	3	0	0

- The second tableau is already optimal, but the artificial y_2 remains in the basis.

- We cannot pivot again in order to eliminate y_2, since in the second row all the coefficients corresponding to non-basic variables from the original problem (namely, y_2) are zero.
- In this case we simply remove the row corresponding to variable y_2.
- Then, we remove the artificial variables and restate the original objective function.

Table: First Simplex Tableau - Phase II.

	x_1	x_2	RHS
x_1	1	1	2
z'	0	1	-2

- From now on we can start the phase II (left as an exercise).

- An example which shows the infeasability of the original problem:

$$\begin{aligned} \text{minimize} \quad & z = -x_1 \\ \text{subject to} \quad & \\ & x_1 + x_2 \geqslant 6 \\ & 2x_1 + 3x_2 \leqslant 4 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

- We convert to the standard form, add artificial variables and modify the objective function

$$\begin{aligned} \text{minimize} \quad & z = y_1 \\ \text{subject to} \quad & \\ & x_1 + x_2 - x_3 + y_1 = 6 \\ & 2x_1 + 3x_2 + x_4 = 4 \\ & x_1, x_2, x_3, x_4, y_1 \geqslant 0 \end{aligned}$$

Table: First Simplex Tableau - Phase I.

	x_1	x_2	x_3	x_4	y_1	RHS		
y_1	1	1	-1	0	1	6	6/1	
x_4	2	3	0	1	0	4	4/2	← min
z'	-1	-1	1	0	0	-6		

Table: Second Simplex Tableau - Phase I.

	x_1	x_2	x_3	x_4	y_1	
y_1	0	-1/2	-1	-1/2	1	4
x_1	1	3/2	0	1/2	0	2
z'	0	1/2	1	1/2	0	-4

- The Phase I problem has a non-zero optimum value, hence the original problem in infeasible. We must stop here - there is no Phase II problem.

- Historically, the big M method precedes the two phase method; it has been replaced due to the grater practical efficiency of the former.
- The big M method ensures that the artificial variables are zero in an (or, equivalently, in any) optimal feasible solution.
- That is, it pushes the artificial variables out of the optimal basis, by assigning a penalty cost M to each such variable in the objective function, where $M >$ is a big real number.
- Hence, instead of the original problem, we will solve

$$\begin{aligned} \text{minimize} \quad & z' = \mathrm{c}^t \mathrm{x} + \sum_j M y_j, \\ \text{subject to} \quad & \mathrm{Ax} + \mathrm{y} = \mathrm{b}, \\ & \mathrm{x}, \mathrm{y} \geqslant 0. \end{aligned} \tag{5}$$

- Problem (2) has feasible solutions if and only if there is an optimal feasible solution of (5) having $y = 0$.
- A basic feasible solution for (2) can be derived from an optimal solution to (5) in a similar manner to that of two phase method.
- Obviously, in order to solve (5), having the artificial variables as the initial basis, we must eliminate all of them from the objective function:

$$y_j = b_j - \sum_i a_{ji} x_i, \forall j$$

$$z' = \sum_i \left(c_i - M \sum_j a_{ji} \right) x_i + M \sum_j b_j.$$

- As for the Two Phase method, we don't add artificial variables to those constraints who already have slack variables.

- We will use again the following example

$$\begin{aligned} \text{minimize} \quad & z = 2x_1 + 3x_2 \\ \text{subject to} \quad & \\ & 3x_1 + 2x_2 = 14 \\ & 2x_1 - 4x_2 \geqslant 2 \\ & 4x_1 + 3x_2 \leqslant 19 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

- In standard form the problem becomes

$$\begin{aligned} \text{minimize} \quad & z = 2x_1 + 3x_2 \\ \text{subject to} \quad & \\ & 3x_1 + 2x_2 = 14 \\ & 2x_1 - 4x_2 - x_3 = 2 \\ & 4x_1 + 3x_2 + x_4 = 19 \\ & x_1, x_2, x_3, x_4 \geqslant 0 \end{aligned}$$

- We add artificial variables

$$
\begin{aligned}
\text{minimize} \quad & z = 2x_1 + 3x_2 + My_1 + My_2 \\
\text{subject to} \quad & \\
& 3x_1 + 2x_2 + y_1 = 14 \\
& 2x_1 - 4x_2 - x_3 + y_2 = 2 \\
& 4x_1 + 3x_2 + x_4 = 19 \\
& x_1, x_2, x_3, x_4, y_1, y_2 \geqslant 0
\end{aligned}
$$

- Or, after we eliminate y_1 and y_2

$$
\begin{aligned}
\text{minimize} \quad & z = (2 - 5M)x_1 + (3 + 2M)x_2 + Mx_3 + 16M \\
\text{subject to} \quad & \\
& 3x_1 + 2x_2 + y_1 = 14 \\
& 2x_1 - 4x_2 - x_3 + y_2 = 2 \\
& 4x_1 + 3x_2 + x_4 = 19 \\
& x_1, x_2, x_3, x_4, y_1, y_2 \geqslant 0
\end{aligned}
$$

Table: First Simplex Tableau.

	x_1	x_2	x_3	x_4	y_1	y_2	RHS	
y_1	3	2	0	0	1	0	14	14/3
y_2	2	-4	-1	0	0	1	2	1/2 ← min
x_4	4	3	0	1	0	0	19	19/2
z'	$2-5M$	3+2M	M	0	0	0	-16M	

Table: Second Simplex Tableau.

	x_1	x_2	x_3	x_4	y_1	y_2	RHS	
y_1	0	8	3/2	0	1	-3/2	11	
x_1	1	-2	-1/2	0	0	1/2	1	
x_4	0	11	2	1	0	-2	15	15/11 ←
z'	0	$7-8M$	1 - 3/2M	0	0	-1 + 5/2M	-2 - 11M	

Table: Third Simplex Tableau.

	x_1	x_2	x_3	x_4	y_1	y_2	RHS		
y_1	0	0	1/22	-8/11	1	-1/22	1/11	2	← min
x_1	1	0	-3/22	2/11	0	3/22	41/11		
x_2	0	1	2/11	1/11	0	-2/11	15/11	15/2	
z'	0	0	$-\frac{M+6}{22}$	$\frac{8M-7}{11}$	0	$\frac{6-23M}{22}$	$-\frac{127+M}{11}$		

Table: Fourth Simplex Tableau.

	x_1	x_2	x_3	x_4	y_1	y_2	RHS
x_3	0	0	1	-16	22	-1	2
x_1	1	0	0	-2	3	0	4
x_2	0	1	0	3	-4	0	1
z'	0	0	0	-5	M + 6	-12M/11	-11

- Although not optimal, the current basis doesn't contain any artificial variable, so this is a basic feasible solution for the original problem.
- We remove the artificial variable and restate the original objective function: $z = 2x_1 + 3x_2 = 11 - 5x_4$.

Table: Modified Simplex Tableau.

	x_1	x_2	x_3	x_4	RHS		
x_3	0	0	1	-16	2		
x_1	1	0	0	-2	4		
x_2	0	1	0	3	1	1/3	← min
z'	0	0	0	-5	-11		

- From this point we can use the simplex algorithm for the original problem ...

Bibliography

- Bertsimas, D., J. N. Tsitsiklis, *Introduction to Linear Optimization*, Athena Scientific, Belmont, Massachusetts, 1997.
- Griva, I., S. G. Nash, A. Sofer, *Linear and Nonlinear Optimization*, 2nd edition, SIAM, 2009.
- Kolman, B., R. E. Deck, *Elementary Linear Programming with Applications*, Elsevier Science and Technology Books, 1995.
- Taha, H. A., *Operations Research: An Introduction*, Prentice Hall International, 8th edition, 2007.