Operations Research - Lecture 3

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Consider an LP problem in standard form

$$\begin{aligned} \text{minimize} \quad & z = c^T x + d, \\ \text{subject to} \quad & Ax = b, \\ & x \geqslant 0. \end{aligned} \tag{1}$$

Theorem

If an LP problem in standard form has a finite optimal solution (i. e., the problem is bounded), then it has an optimal basic feasible solution.

Proof. From the Representation Theorem, we know that, for some $v^1, v^2, \dots, v^t \in \mathbb{E}_{\mathcal{P}}$, and y (a direction of unboundedness or a zero vector), we have

$$x = y + \sum_{i=1}^{t} \lambda_i v^i, \text{ where } \sum_{i=1}^{t} \lambda_i = 1, \lambda_i \geqslant 0, \forall i = \overline{1, t}.$$

Let x be an optimal solution. Since we know that $x + \alpha y \in \mathcal{P}$, for all $\alpha > 0$[1], we must have

$$c^T x \leqslant c^T(x + \alpha y) \Longrightarrow \alpha c^T y \geqslant 0 \Longrightarrow c^T y \geqslant 0.$$

Suppose that $c^T y > 0$; if we denote $\widetilde{x} = x - y \in \mathcal{P}$ (check!), then $c^T x > c^T \widetilde{x}$, which implies that x would not be optimal. Therefore, $c^T y = 0$, and $c^T x = c^T \widetilde{x}$, that is, $\widetilde{x}$ is also an optimal solution. On the other hand

$$c^T \widetilde{x} = \sum_{i=1}^{t} \lambda_i c^T v^i \geqslant \sum_{i=1}^{t} \lambda_i c^T \widetilde{x} = c^T \widetilde{x} \Longrightarrow c^T v^i = c^T \widetilde{x}, \forall i = \overline{1, t},$$

Hence any v^j, with $\lambda_j > 0$, is an optimal basic feasible solution for problem (1). (v^j being an extreme point of $\mathcal{P}$ it corresponds to a basic feasible solution.) □

[1] Recall the definition of a direction of unboundedness.

Finite Optimal Solution

- A *feasible direction* for x is a vector $y \in \mathbb{R}^n$ satisfying

$$Ay = 0 \text{ and } x_i = 0 \Rightarrow y_i = 0.$$

- Suppose now that x is a finite optimal basic feasible solution and y is a feasible direction of the same problem. For small enough $\varepsilon > 0$, $(x + \varepsilon y)$ must be a feasible solution too.
- Under these conditions y must satisfy

$$c^T y \geqslant 0, Ay = 0, \text{ and } x_i = 0 \Rightarrow y_i = 0.$$

- Simplex method was developed in the 1940's, in the research department of US Air Force, when the linear programming models show their appeal for military and later economic planning (as a matter of fact linear programming means linear planning).
- This method benefited from the almost simultaneous development of digital programmable computers, which gave tools for automated solving of large scale linear problems.
- Over the years simplex algorithm has proved its efficiency and was the main LP method until the 1980's when another LP technique was discovered (namely, the interior path method).
- Even today simplex remains a prominent method, partly because of its simplicity (sic!) and because of its theoretical applications.

- We already know from the previous section that, if an LP problem in standard form has a finite optimal solution, then it has an optimal basic feasible one.
- In other words, if an LP problem has a finite optimum, then an optimum feasible solution may be found among the extreme points of the subjacent polyhedra.
- *Simplex algorithm* is based on these observations and searches for an optimal feasible solution by moving from one basic feasible solution to another (adjacent one), along the boundary of the feasible region, improving the objective function.
- Eventually a basic feasible solution is reached at which no (guaranteed) improvements of the objective function are possible. Such a basic feasible solution is an optimal one.

Algebra of Simplex

- We consider here the LP problem (1) in standard form (remember that $b \geqslant 0$). Let x be a basic feasible solution with the variables ordered so that
$$x^T = (x_B^T \; x_N^T),$$
where x_B is the vector of basic variables and x_N is the vector of non-basic ones ($x_N = 0$).
- Correspondingly we split c and A:
$$c^T = (c_B^T \; c_N^T), A = (B \; N).$$
- The objective function and the constraints become
$$z = c_B^T x_B + c_N^T x_N, \quad B x_B + N x_N = b.$$
- From the last equalities we get
$$x_B = B^{-1} b - B^{-1} N x_N \Rightarrow z = c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N) x_N.$$

- If we take $\mathrm{y} = (\mathrm{c}_B^T \mathrm{B}^{-1})^T$, the objective function become

$$z = \mathrm{y}^T \mathrm{b} + (\mathrm{c}_N^T - \mathrm{y}^T \mathrm{N}) \mathrm{x}_N .$$

- y is the vector of *simplex multipliers*.
- The current values of basic variables and objective function are

$$\mathrm{x}_B = \widehat{\mathrm{b}} = \mathrm{B}^{-1}\mathrm{b}, \;\; \widehat{z} = \mathrm{y}^T \mathrm{b}.$$

We consider an example already analyzed

$$
\begin{array}{ll}
\text{minimize} & z = -5x_1 - 4x_2 \\
\text{subject to} & \\
& \begin{aligned}
6x_1 + 4x_2 + x_3 &= 24 \\
x_1 + 2x_2 + x_4 &= 6 \\
-x_1 + x_2 + x_5 &= 1 \\
x_2 + x_6 &= 2 \\
x_1, x_2, \ldots, x_6 &\geqslant 0
\end{aligned}
\end{array}
$$

We have $c = (-5, -4, 0, 0, 0, 0)^T$, $b = (24, 6, 1, 2)^T$, and

$$
A = \begin{bmatrix} 6 & 4 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.
$$

- Consider the feasible base $\{x_1, x_4, x_5, x_6\}$, $\mathrm{x}_B = (x_1\ x_4\ x_5\ x_6)^T$, $\mathrm{x}_N = (x_2\ x_3)^T$, and

$$\mathrm{B} = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathrm{B}^{-1} = \begin{bmatrix} 1/6 & 0 & 0 & 0 \\ -1/6 & 1 & 0 & 0 \\ 1/6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathrm{N} = \begin{bmatrix} 4 & 1 \\ 2 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix},$$

$\mathrm{c}_B = (-5\ 0\ 0\ 0)^T$, $\mathrm{c}_N = (-4\ 0)^T$.

- We may further compute

$$\mathrm{y} = (-5/6\ 0\ 0\ 0)^T, \mathrm{x}_B = \widehat{\mathrm{b}} = (4\ 2\ 5\ 2)^T, \widehat{z} = -28.$$

- The general formula for the objective value is

$$z = y^T b + (c_N^T - y^T N) x_N.$$

Definition

The coefficient corresponding to a non-basic variable x_j in the vector $\widehat{c}_N^T = (c_N^T - c_B^T B^{-1} N)$ is called the reduced cost of variable *x_j:*

$$\widehat{c}_j = c_j - c_B^T B^{-1} A_j \quad (2)$$

- If we want to test for optimality we examine, in (2), the variation of the objective function if a non-basic variable x_j is increased from zero: if $\widehat{c}_j < 0$, then the objective will decrease, if $\widehat{c}_j > 0$, then the objective will increase, and if $\widehat{c}_j = 0$, then the objective will not change.

Theorem

Let x *be a basic feasible solution to the LP problem (1), having the basis matrix* B, *and* $\widehat{\mathrm{c}}_N$ *the vector of reduced costs for non-basic variables. The following are true*

(i) *If* $\widehat{\mathrm{c}}_N \geqslant 0$, *then* x *is an optimal solution.*

(ii) *If* x *is an optimal and nondegenerate solution, then* $\widehat{\mathrm{c}}_N \geqslant 0$.

Proof. (i) Let $\widetilde{\mathrm{x}}$ be an arbitrary feasible solution of (1), and $\overline{\mathrm{x}} = \widetilde{\mathrm{x}} - \mathrm{x}$. We have

$$\mathrm{A}\mathrm{x} = \mathrm{A}\widetilde{\mathrm{x}} = \mathrm{b} \Rightarrow \mathrm{A}\overline{\mathrm{x}} = 0 \Rightarrow \mathrm{B}\overline{\mathrm{x}}_B + \sum_{j\in N} \mathrm{A}_j \overline{x}_j = 0.$$

From here we get $\overline{\mathrm{x}}_B = -\sum_{j\in N} \mathrm{B}^{-1}\mathrm{A}_j \overline{x}_j$, and

$$c^T\overline{x} = c_B^T\overline{x}_B + \sum_{j\in N} c_j\overline{x}_j = \sum_{j\in N}(c_j - c_B^T B^{-1}A_j)\overline{x}_j = \sum_{j\in N}\widehat{c}_j\overline{x}_j =$$

$$\sum_{j\in N}(\widehat{c}_j\widetilde{x}_j - \widehat{c}_j x_j) = \sum_{j\in N}\widehat{c}_j\widetilde{x}_j \geqslant 0.$$

Thus, $c^Tx \leqslant c^T\widetilde{x}$, for any feasible solution $\widetilde{x}$, which means that x is an optimal solution.

(ii) Let x be a nondegenerate (that is, $x_i > 0, \forall i \in B$) optimal solution and suppose that we have $\widehat{c}_j < 0$, for some $j \in N$.

We can build a feasible solution $x + \alpha y$ ($\alpha > 0$), such that the j-th (non-basic) variable is increased, and all other non-basic variables remain equal with zero: $y_j = 1$, and $y_h = 0, \forall h \in N \setminus \{j\}$.

$$A(x+\alpha y) = b \Rightarrow Ay = 0 \Rightarrow 0 = \sum_{i=1}^{n} A_i y_i = \sum_{i\in B} A_i y_i + A_j = By_B + A_j.$$

Algebra of Simplex

From the above relations we get $y_B = -B^{-1}A_j$; with these values y is the *j-th basic direction*.

Obviously, for small enough $\alpha > 0$, we will have $(x + \alpha y) > 0$. Now,

$$c^T y = c_B^T y_B + c_j y_j = -c_B^T B^{-1} A_j + c_j y_j = \widehat{c}_j.$$

Since $\widehat{c}_j < 0$, we have $c^T(x + \alpha y) = c^T x + \alpha \widehat{c}_j < c^T x$, hence x cannot be an optimal solution - a contradiction. We must have $\widehat{c}_N \geqslant 0$. □

- Using Theorem 3.1, we acknowledge that, if x is a basic feasible solution, and $\widehat{c}_j < 0$, we can eventually increase x_j (this is the *entering variable*) until a nonnegativity constraint is violated (this will give the *leaving variable*).
- The basic variables are $x_B = B^{-1}b - B^{-1}N x_N$ and all components of x_N are zero, except x_j. Therefore,

$$x_B = \widehat{b} - \widehat{A}_j x_j, \text{ where } \widehat{A}_j = B^{-1} A_j. \tag{3}$$

- We examine equation (3) componentwise: $x_i = \widehat{b}_i - \widehat{a}_{ij} x_j$, $i \in B$:
  - if $\widehat{a}_{ij} > 0$, then x_i will decrease as x_j increases and will become zero when $x_j = \dfrac{\widehat{b}_i}{\widehat{a}_{ij}}$;
  - if $\widehat{a}_{ij} < 0$, then x_i will increase as x_j increases (hence x_j remains non-negative);
  - if $\widehat{a}_{ij} = 0$, then x_i will have the same (non-negative) value.
- The variable x_j can be increased as long as all variables have non-negative values:

$$\widehat{x}_j = \min \left\{ \frac{\widehat{b}_i}{\widehat{a}_{ij}} : \widehat{a}_{ij} > 0 \right\}. \tag{4}$$

- What happens if $\widehat{a}_{ij} \leqslant 0, \forall i \in B$? The answer is given by the following result.

Theorem

Let x be a basic feasible solution to the LP problem (1), having the basis matrix B, and $\widehat{\mathrm{c}}_N$ the vector of reduced costs for non-basic variables. Suppose that $\widehat{c}_j < 0$, for some $j \in N$; if $\widehat{a}_{ij} \leqslant 0$, for all $i \in B$, then problem (1) has an infinite optimum (it is unbounded).

Proof. Obviously, all the basic variables will not decrease (see the above theorem), and x_j can be made arbitrary large. The new values of the basic variables and of the objective function are

$$\mathrm{x}_B \leftarrow \mathrm{x}_B - \mathrm{A}_j \widehat{x}_j, \; z = \widehat{z} + \widehat{c}_j \widehat{x}_j \quad (5)$$

We have $\lim\limits_{\widehat{x}_j \to +\infty} \widehat{z} = -\infty$, this is the "optimum" of the problem (which is unbounded). □

The algorithm starts with with a basis matrix B, corresponding to the basic feasible solution $\mathrm{x}_B = \widehat{\mathrm{b}} = \mathrm{B}^{-1}\mathrm{b} \geqslant 0$. The algorithm follows:

The Optimality Test. Compute $\mathrm{y}^T = \mathrm{c}_B^T \mathrm{B}^{-1}$ and $\widehat{\mathrm{c}}_N^T = \mathrm{c}_N^T - \mathrm{y}^T \mathrm{N}$; if $\widehat{\mathrm{c}}_N^T \geqslant 0$, then the current base is optimal, if not, select an index $j \in N$, such that $\widehat{c}_j < 0$. x_j will be the *entering variable*.

The Main Step. Compute $\widehat{\mathrm{A}}_j = \mathrm{B}^{-1}\mathrm{A}_j$. If $\widehat{a}_{hj} \leqslant 0$, for all $h \in B$, then Stop - the problem has infinite optimum. Otherwise find an $i \in B$ such that

$$\frac{\widehat{b}_i}{\widehat{a}_{ij}} = \min\left\{\frac{\widehat{b}_h}{\widehat{a}_{hj}} : \widehat{a}_{hj} > 0\right\}.$$

x_i will be the *leaving variable* and $\widehat{a}_{ij}$ will be the *pivot* entry.

The Update. Compute the new basis matrix B, the new vector of basic variables x_B, and the new reduced costs $\widehat{c}$. Go to the *optimality test*.

The Simplex Algorithm - an Example

- We consider again our example

$$A = \begin{bmatrix} 6 & 4 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \mathrm{b} = \begin{bmatrix} 24 \\ 6 \\ 1 \\ 2 \end{bmatrix}, \text{ and } \mathrm{c} = \begin{bmatrix} -5 \\ -4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

- But now we will use the slack variables as an initial basic feasible solution: $\{x_3, x_4, x_5, x_6\}$. Hence $\mathrm{x}_B = (x_3\ x_4\ x_5\ x_6)^T$, $\mathrm{x}_N = (x_1\ x_2)^T$, $\mathrm{B} = I_4 = \mathrm{B}^{-1}$, $\mathrm{c}_B^T = (0\ 0\ 0\ 0)$, $\mathrm{c}_N^T = (-5\ \ -4)$, and

$$N = \begin{bmatrix} 6 & 4 \\ 1 & 2 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

- We compute the basis: $\mathrm{x}_B = \widehat{\mathrm{b}} = \mathrm{B}^{-1}\mathrm{b} = (24\ \ 6\ \ 1\ \ 2)^T$; then

$$\mathrm{y}^T = \mathrm{c}_B^T \mathrm{B}^{-1} = (0\ \ 0\ \ 0\ \ 0),\ \widehat{\mathrm{c}}_N^T = \mathrm{c}_N^T - \mathrm{y}^T\mathrm{N} = (-5\ \ -4).$$

- The current basis is not optimal because $\widehat{\mathrm{c}}_N$ has negative components; we choose $\widehat{c}_1 < 0$ - hence, x_1 will be the *entering varable*. We also compute

$$\widehat{\mathrm{A}}_1 = \mathrm{B}^{-1}\mathrm{A}_1 = (6\ 1\ -1\ 0)^T$$

- In order to find out the *leaving variable* we apply the ratio test:

$$\frac{\widehat{b}_3}{\widehat{a}_{31}} = 4, \frac{\widehat{b}_4}{\widehat{a}_{41}} = 6 \ (\widehat{a}_{51} < 0, \widehat{a}_{61} = 0).$$

- x_3 will be the leaving variable;
- In the next iteration x_1 will replace x_3 in the new basis: $\mathrm{x}_B = (x_1 \ x_4 \ x_5 \ x_6)^T, \mathrm{x}_N = (x_3 \ x_2)^T,$

$$\mathrm{B} = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathrm{B}^{-1} = \begin{bmatrix} 1/6 & 0 & 0 & 0 \\ -1/6 & 1 & 0 & 0 \\ 1/6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathrm{N} = \begin{bmatrix} 1 & 4 \\ 0 & 2 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\mathrm{c}_B = (-5 \ 0 \ 0 \ 0)^T, \mathrm{c}_N = (0 \ -4)^T.$$

- Thus

$$x_B = \widehat{b} = B^{-1}b = (4\ \ 2\ \ 5\ \ 2)^T,$$

$$y^T = c_B^T B^{-1} = (-5/6\ \ 0\ \ 0\ \ 0),$$

$$\widehat{c}_N^T = c_N^T - y^T N = (5/6\ \ -4/6).$$

- The second reduced cost is negative, hence the current base is not optimal, x_2 will be the next the entering variable ...

- The tableaux are a convenient and compact form to present the simplex algorithm; they are just a notational tool.
- In this format the inverse of basis matrix are updated at every iteration and not computed anew, increasing the speed of the method.
- The original LP problem corresponds to the tableau

	$\mathbf{x}_B$	$\mathbf{x}_N$	RHS
$\mathbf{x}_B$	$\mathbf{B}$	$\mathbf{N}$	$\mathbf{b}$
z	$\mathbf{c}_B^T$	$\mathbf{c}_N^T$	0

- The tableau for the problem with respect to the current basis is

	$\mathbf{x}_B$	$\mathbf{x}_N$	RHS
$\mathbf{x}_B$	I_m	$\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{B}^{-1}\mathbf{b}$
z	0	$\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N}$	$\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}$

- We consider the following problem

$$\begin{array}{ll} \text{minimize} & z = -10x_1 - 12x_2 - 12x_3 \\ \text{subject to} & \\ & \begin{aligned} x_1 + 2x_2 + 2x_3 &\leqslant 20 \\ 2x_1 + x_2 + 2x_3 &\leqslant 20 \\ 2x_1 + 2x_2 + x_3 &\leqslant 20 \\ x_1, x_2, \ldots, x_3 &\geqslant 0 \end{aligned} \end{array}$$

- In standard form, the problem becomes

$$\begin{array}{ll} \text{minimize} & z = -10x_1 - 12x_2 - 12x_3 \\ \text{subject to} & \\ & \begin{aligned} x_1 + 2x_2 + 2x_3 + x_4 &= 20 \\ 2x_1 + x_2 + 2x_3 + x_5 &= 20 \\ 2x_1 + 2x_2 + x_3 + x_6 &= 20 \\ x_1, x_2, \ldots, x_6 &\geqslant 0 \end{aligned} \end{array}$$

- Observe that $\mathrm{x} = (0\ 0\ 0\ 20\ 20\ 20)^T$ is a basic feasible solution and it can start the algorithm.

Table: First Simplex tableau (note that $B = I_3$).

	x_1	x_2	x_3	x_4	x_5	x_6	RHS		
x_4	1	2	2	1	0	0	20	20/1	
x_5	2	1	2	0	1	0	20	20/2	← min
x_6	2	2	1	0	0	1	20	20/2	← min
z	-10	-12	-12	0	0	0	0		

- The reduced cost of x_1 is negative: we let this variable to *enter the basis*; the *pivot column* is that labeled by x_1.
- The smallest ratio corresponds to either the row labeled by x_5 or by x_6; we choose the former. That will be the *pivot row*; x_5 will *leave the basis*.

- The process of updating the tableau is called *pivoting*: add to each row of the tableau a multiple of the pivot row such that the pivot element becomes 1 and all other entries of the pivot column become 0.
- We apply this rule of transformation to our tableau
  - multiply the pivot row by -0.5 and add it to the first row;
  - substract the pivot row from the third row;
  - multiply the pivot row by 5 and add it to the last row;
  - finally we divide the pivot row by 2.
- The tableau becomes (x_1 will now label the second row):

Table: Second Simplex tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_4	0	1.5	1	1	-0.5	0	10
x_1	1	0.5	1	0	0.5	0	10
x_6	0	1	-1	0	-1	1	0
z	0	-7	-2	0	5	0	100

- The current basis is $\{x_1, x_4, x_6\}$ - not an optimal one, since the non-basic variables x_2 and x_3 have negative reduced costs. We choose x_3 to be the *entering variable*.
- Minimum ratio corresponds to both first and second rows; we choose x_4 to be the *leaving variable*.
- The position of the *pivot* is at the intersection of the first row with the third column.

Table: Second Simplex tableau decorated.

	x_1	x_2	x_3	x_4	x_5	x_6	RHS		
x_4	0	1.5	1	1	-0.5	0	10	10/1	← min
x_1	1	0.5	1	0	0.5	0	10	10/1	← min
x_6	0	1	-1	0	-1	1	0		
z	0	-7	-2	0	5	0	100		

- We pivot again:
  - substract the pivot row from the second row;
  - add the pivot row to the third row;
  - multiply by 2 the pivot row and add it to the last row;
  - the pivot row remains unchanged (since the pivot has value 1).

Table: Third Simplex tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	0	1.5	1	1	-0.5	0	10
x_1	1	-1	0	-1	1	0	0
x_6	0	2.5	0	1	-1.5	1	10
z	0	-4	0	2	4	0	120

- The current basis is $\{x_1, x_3, x_6\}$ - not an optimal one, since the non-basic variable x_2 has negative reduced cost; x_2 will be the *entering variable*.
- Minimum ratio corresponds to both third row; we choose x_6 to be the *leaving variable*.
- The position of the *pivot* is at the intersection of the third row with the second column.

Table: Third Simplex tableau decorated.

	x_1	x_2	x_3	x_4	x_5	x_6	RHS		
x_3	0	1.5	1	1	-0.5	0	10	10/1.5	
x_1	1	-1	0	-1	1	0	0		
x_6	0	**2.5**	0	1	-1.5	1	10	10/2.5	← min
z	0	-4	0	2	4	0	120		

- We pivot again:
  - multiply by -0.6 the pivot row and add it to the first row;
  - multiply by 0.4 the pivot row and add it to the second row;
  - multiply by 1.6 the pivot row and add it to the last row;
  - divide the pivot row by 2.5.

Table: Fourth Simplex tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	0	0	1	0.4	0.4	-0.6	4
x_1	1	0	0	-0.6	0.4	0.4	4
x_2	0	1	0	0.4	-0.6	0.4	4
z	0	0	0	3.6	1.6	1.6	136

- The current basis - $\{x_1, x_2, x_3\}$ - is an optimal one, since all nonbasic variables have nonnegative reduced costs.
- We find out an optimal solution: $x_1 = x_2 = x_3 = 4$, the optimal value of the objective function is -136 (don't forget to change the sign).

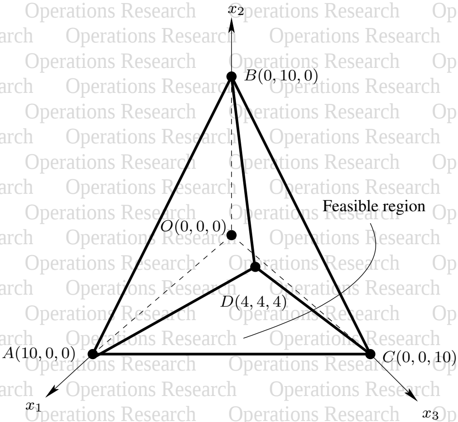

Figure: Feasible Region of our example

- If compare the walk of our simplex tableaus with the graphic of the feasible region we see that this corresponds to the path O, A, C, D.
- Obviously, if we choose other variables to enter or to leave (when this is possible), we could find another path through the extreme points of the feasible region.
- However, some paths are not eligible for the simplex algorithm: path O, A, D could not be traced, since the initial and the final bases differ by three variables (at least three basis changes are required).

- Obviously, an LP problem can have more than one optimal solution: such a problem can have one, none or an infinite set of optimal solutions.
- We can detect such situations: when, for an optimal basic feasible solution x, one of the non-basic variables, $x_j, j \in N$, has zero reduced cost: $\widehat{c}_j = 0$.
- If we let x_j enter the current base, the new base will give the same value for the objective function. Hence, we have another optimal solution; by, say, geometric reasons, this imply that we will have an infinite number of solutions.
- It is easy to check that, if $\mathrm{x}^1, \mathrm{x}^2$ are optimal solution for a LP problem, then any vector of the line segment joining x^1 and x^2 is an optimal solution too.

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