Operations Research - Lecture 2

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Canonical Form

Definition

An LP in canonical form *will be written as*

$$\begin{aligned} \textit{minimize} \quad & z = \sum_{j=1}^{n} c_j x_j + d, \\ \textit{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \geqslant b_i, i = \overline{1, m}, \\ & x_j \geqslant 0, j = \overline{1, n} \end{aligned} \tag{1}$$

Sometimes the constant from the objective function is dropped (being considered 0 does not modify the optimal solution, but only the value of the objective).

In matricial notations (1) becomes

$$\begin{aligned} \text{minimize} \quad & z = \mathrm{c}^T \mathrm{x} + d, \\ \text{subject to} \quad & \mathrm{A}\mathrm{x} \geqslant \mathrm{b}, \\ & \mathrm{x} \geqslant 0. \end{aligned} \tag{2}$$

where $\mathrm{b} \in \mathbb{R}^m$, $\mathrm{c} \in \mathbb{R}^n$, and $\mathrm{A} \in \mathbb{R}^{m \times n}$ is the *constraint matrix*. A LP problem in canonical form has the following features:

- is a minimization problem;
- all the variables are restricted to be non-negative;
- all other constraints are "$\geqslant$" inequations.

Standard Form

Definition

An LP in standard form *will be written as*

$$\begin{aligned} \text{minimize} \quad & z = \sum_{j=1}^{n} c_j x_j + d, \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i, \, i = \overline{1, m}, \\ & x_j \geqslant 0, j = \overline{1, n} \end{aligned} \tag{3}$$

where $b_j \geqslant 0$, *for* $j = \overline{1, m}$.

Standard Form

In matricial notations

$$\begin{aligned} \text{minimize} \quad & z = \mathrm{c}^T \mathrm{x} + d, \\ \text{subject to} \quad & \mathrm{A}\mathrm{x} = \mathrm{b}, \\ & \mathrm{x} \geqslant 0. \end{aligned} \tag{4}$$

where $\mathrm{b} \geqslant 0$. An LP problem in standard form has the following features:

- is a minimization problem;
- all the variables are restricted to be non-negative;
- all other constraints are equations;
- the components of the right-hand side vector b are non-negative.

Any LP problem can be converted to standard form; we illustrate the techniques of converting using some examples.

- If we have a maximization problem:

$$\text{maximize } z = \mathrm{c}^T \mathrm{x} + d$$

we multiply the objective by (-1):

$$\text{minimize } z' = -\mathrm{c}^T \mathrm{x} - d$$

After we solve the problem, the optimal objective value must be multiplied by (-1): $z_* = -z'_*$.
However, the optimal solutions (that is, the values of the variables after solving the problem) are the same.

- A (non-zero) lower bound on a variable:

$$x_h \geqslant b_h, \text{ with } b_h \neq 0.$$

is treated by replacing the variable with:

$$x'_h = x_h - b_h \text{ and } x'_h \geqslant 0.$$

- Upper bounds of a variable can be treated in a similar manner or as a general constraint.

- An inequation having a negative right-hand side:

$$a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jn}x_n \leqslant b_j \text{ (or } \geqslant b_j\text{)}, \text{ with } b_j < 0.$$

must be multiplied by (-1):

$$-a_{j1}x_1 - a_{j2}x_2 - \ldots - a_{jn}x_n \geqslant -b_j \text{ (or } \leqslant -b_j\text{)}.$$

- An unrestricted variable:

$$x_h \in \mathbb{R}$$

can be replaced by a pair of non-negative variables like this

$$x_h = x_h' - x_h''.$$

- An "$\leqslant$" inequation with a non-negative right-hand side:

$$a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jn}x_n \leqslant b_j, \text{ with } b_j \geqslant 0.$$

is converted to an equation by including a *slack variable* $s_j \geqslant 0$

$$a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jn}x_n + s_j = b_j.$$

- An "$\geqslant$" inequation with a non-negative right-hand side:

$$a_{h1}x_1 + a_{h2}x_2 + \ldots + a_{hn}x_n \geqslant b_h, \text{ with } b_h \geqslant 0.$$

is converted to an equation by including an *excess variable* $e_h \geqslant 0$

$$a_{h1}x_1 + a_{h2}x_2 + \ldots + a_{hn}x_n - e_h = b_h.$$

$$
\begin{array}{ll}
\text{maximize} & z = 3x_1 - 2x_2 + 4x_3 + 2 \\
\text{subject to} & \\
& \begin{aligned} x_1 + 4x_2 - x_3 &\leqslant 10 \\ -x_1 + 8x_2 + 2x_3 + x_4 &\geqslant 1 \\ x_1 - 3x_2 - x_4 &= 4 \end{aligned} \\
& x_1 \geqslant 3, 0 \leqslant x_3 \leqslant 5
\end{array}
$$

First we convert the variables and the objective function:

$$
\begin{array}{ll}
\text{minimize} & z' = -3x_1' + 2x_2' - 2x_2'' - 4x_3 + 11 \\
\text{subject to} & \\
& \begin{aligned} x_1' + 4x_2' - 4x_2'' - x_3 &\leqslant 7 \\ -x_1' + 8x_2' - 8x_2'' + 2x_3 + x_4' - x_4'' &\geqslant 4 \\ x_1' - 3x_2' + 3x_2'' - x_4' + x_4'' &= 1 \\ x_3 &\leqslant 5 \end{aligned} \\
& x_1', x_2', x_2'', x_3, x_4', x_4'' \geqslant 0
\end{array}
$$

Then we convert the constraints:

$$\text{minimize} \quad z' = -3x_1' + 2x_2' - 2x_2'' - 4x_3 + 11$$

subject to

$$\begin{aligned} x_1' + 4x_2' - 4x_2'' - x_3 + s_1 &= 7 \\ -x_1' + 8x_2' - 8x_2'' + 2x_3 + x_4' - x_4'' - e_2 &= 4 \\ x_1' - 3x_2' + 3x_2'' - x_4' + x_4'' &= 1 \\ x_3 + s_3 &= 5 \\ x_1', x_2', x_2'', x_3, x_4', x_4'', s_1, e_2, s_3 &\geqslant 0 \end{aligned}$$

In matrix notations the problem is

$$\begin{aligned} \text{minimize} \quad & z = \mathrm{c}^T \mathrm{x} + d, \\ \text{subject to} \quad & \mathrm{A}\mathrm{x} = \mathrm{b}, \\ & \mathrm{x} \geqslant 0. \end{aligned} \tag{5}$$

where $d = 11, \mathrm{c} = (-3, 2, -2, -4, 0, 0, 0, 0, 0)^T, \mathrm{b} = (7, 4, 1, 5)^T$, and

$$\mathrm{A} = \begin{bmatrix} 1 & 4 & -4 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 8 & -8 & 2 & 1 & -1 & 0 & -1 & 0 \\ 1 & -3 & 3 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Definition

Let $\mathrm{a} \in \mathbb{R}^n$ *be a non-zero vector and* $b \in \mathbb{R}$*.*

(i) *The set* $\mathcal{H}_s(\mathrm{a}, b) = \{\mathrm{x} \in \mathbb{R}^n \ : \ \mathrm{x}^T \mathrm{a} \geqslant b\}$ *is called a halfspace.*

(ii) *The set* $H_p(\mathrm{a}, b) = \{\mathrm{x} \in \mathbb{R}^n : \mathrm{x}^T \mathrm{a} = b\}$ *is called a hyperplane.*

- Obviously, a hyperplane $H_p(\mathrm{a}, b)$ is the boundary of two halfspaces: $\mathcal{H}_s(-\mathrm{a}, -b)$ and $\mathcal{H}_s(\mathrm{a}, b)$.
- Geometrically, the vector a is orthogonal on the hyperplane $H_p(\mathrm{a}, b)$; it is called the *normal* vector to $H_p(\mathrm{a}, b)$.

Definition

Let A *be a* $m \times n$ *matrix and* $\mathrm{b} \in \mathbb{R}^n$ *be a vector. The set* $\{\mathrm{x} \in \mathbb{R}^n : \mathrm{Ax} \geqslant \mathrm{b}\}$ *is called a polyhedron.*

- The feasible region of any LP problem is a polyhedron: such a problem can be described by inequality constraints of the form $\mathrm{Ax} \geqslant \mathrm{b}$.
- Obviously a polyhedron is a finite intersection of halfspaces, and can be a bounded or an unbounded set in $\mathbb{R}^n$.

Definition

A set $\mathcal{M} \in \mathbb{R}^n$ *is bounded if it exists a constant* $K \in \mathbb{R}_+^*$, *such that* $||\mathrm{x}||_2 < K$, *for every* $\mathrm{x} \in \mathcal{M}$, *otherwise* $\mathcal{M}$ *is unbounded.*

Definition

A set $\mathcal{M} \in \mathbb{R}^n$ is convex if for any $\mathrm{x}, \mathrm{y} \in \mathcal{M}$, and any $\lambda \in [0,1]$, we have $\lambda \mathrm{x} + (1-\lambda)\mathrm{y} \in \mathcal{M}$.

The *line segment* joining x and y is $\{\mathrm{z} = \lambda \mathrm{x} + (1-\lambda)\mathrm{y} \,:\, \lambda \in [0,1]\}$.

Definition

(i) *A convex combination of vectors $\mathrm{x}^1, \mathrm{x}^2, \dots \mathrm{x}^p \in \mathbb{R}^n$ is a vector $\sum_{i=1}^{p} \lambda_i \mathrm{x}^i$, where $\lambda_1, \lambda_2, \dots, \lambda_p \geqslant 0$ and $\sum_{i=1}^{p} \lambda_i = 1$.*

(ii) *The convex hull, $conv(\mathcal{M})$, of a set $\mathcal{M} \subseteq \mathbb{R}^n$ is the smallest convex set of all convex sets containing $\mathcal{M}$.*

Convexity

Theorem

(i) *An intersection of convex sets is a convex set.*

(ii) *Any of the following sets are convex: a polyhedron, a halfspace, and a hyperplane.*

(iii) *Any convex combination of a finite number of elements of a convex set belongs to that set.*

(iv) *The convex hull of a set* $\mathcal{M} \subseteq \mathbb{R}^n$ *is*

$$conv(\mathcal{M}) = \Big\{ \mathbf{z} \in \mathbb{R}^n \;:\; \exists p \in \mathbb{N}^*, \exists \mathbf{x}^1, \ldots, \mathbf{x}^p \in \mathcal{M},$$

$$and\ \exists \lambda^1, \ldots, \lambda^p \in [0,1], \sum_{i=1}^{p} \lambda_i = 1,\ \ s.t.\ \ \mathbf{z} = \sum_{i=1}^{p} \lambda_i \mathbf{x}^i \Big\}$$

- As we already saw from examples an optimal solution of a LP problem is a "corner" of the polyhedron obtained from all constraints of the problem.
- This is an *extreme point* of a polyhedron: a point that cannot be expressed as a convex combination of other points from that polyhedron.
- The geometric definition follows:

Definition

A vector x *of a set* $\mathcal{M} \subseteq \mathbb{R}^n$ *is an* extreme point *of* $\mathcal{M}$ *if we cannot find two vectors* $\mathrm{y}, \mathrm{z} \in \mathcal{M}$ *and* $\lambda \in (0,1)$ *such that* $\mathrm{x} = \lambda \mathrm{y} + (1-\lambda)\mathrm{z}$. *The* ***set of extreme points*** *of* $\mathcal{M}$ *is denoted by* $\mathbb{E}_{\mathcal{M}}$.

The importance of extreme points in optimization over a polyhedron is revealed by the next result - given here without proof.

Theorem

(Krein-Milman) Any convex, compact subset of $\mathbb{R}^n$ coincides with the convex hull of its extreme points.

Consider now the problem (2) with its subjacent polyhedron $\mathcal{P} = \{\mathrm{x} \in \mathbb{R}^n_+ : \mathrm{Ax} \geqslant \mathrm{b}\}$. If $\mathcal{P}$ is bounded, it is compact (since is obviously closed), hence

Corollary

If $\mathcal{P}$ is bounded, then $\mathcal{P} = conv(\mathbb{E}_{\mathcal{P}})$.

Now consider a LP problem in standard form

$$\begin{aligned} \text{maximize} \quad & z = \mathrm{c}^T \mathrm{x} + d, \\ \text{subject to} \quad & \mathrm{A}\mathrm{x} = \mathrm{b}, \\ & \mathrm{x} \geqslant 0. \end{aligned} \tag{6}$$

where $m \leqslant n$ and matrix $\mathrm{A} \in \mathbb{R}^{m \times n}$ is full rank, that is, its rows are linearly independent. We will see that this condition is not restrictive (see seminar 2).

Definition

A basic solution *is a vector* $\mathrm{x} \in \mathbb{R}^n$ *such that*

(i) x *satisfies the constraints of the linear program:* $\mathrm{A}\mathrm{x} = \mathrm{b}$.

(ii) *The columns of* A *corresponding to non-zero components of* x *are linearly independent.*

Basic Solutions

- The components of a basic solution x can always be separated in two classes: a sub-vector x_N of $(n-m)$ zero components, and x_B of m (possible non-zero) components.
- This separation is possible: since A has full rank we can find a $m \times m$ invertible sub matrix B of A; the columns of B correspond to the variables from x_B also named *basic variables*.
- The set of basic variables is called the *basis* (corresponding to x).
- The variables from x_N are called *non-basic variables*.
- If some of the basic variables are also zero, the above separation could be not unique.

Basic Feasible Solutions

Definition

1. *A basic solution* x *is a* basic feasible solution *if, in addition, it satisfies the non-negativity restrictions, that is,* $x \geqslant 0$.
2. *A basic feasible solution is called* optimal basic feasible solution *if it is also optimal for the linear program.*

Definition

A *feasible solution* *is a vector* $x \in \mathbb{R}^n$ *which satisfies the constraints of the linear program:* $Ax = b$ *and* $x \geqslant 0$.

Consider the Reddy Mikks problem from last course (but as a minimization one)

$$\begin{aligned} \text{minimize} \quad & z = -5x_1 - 4x_2 \\ \text{subject to} \quad & \\ & 6x_1 + 4x_2 \leqslant 24 \\ & x_1 + 2x_2 \leqslant 6 \\ & x_2 - x_1 \leqslant 1 \\ & x_2 \leqslant 2 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

We already solved this problem: optimal value $z_* = -21$ is reached at point $\mathrm{x}_* = (3, 1.5)$ (B). The boundaries of the feasible region are the lines

$$\begin{array}{rclcrclc} 6x_1 + 4x_2 & = & 24 & (1) & x_2 & = & 2 & (4) \\ x_1 + 2x_2 & = & 6 & (2) & x_1 & = & 0 & (5) \\ x_2 - x_1 & = & 1 & (3) & x_2 & = & 0 & (6) \end{array}$$

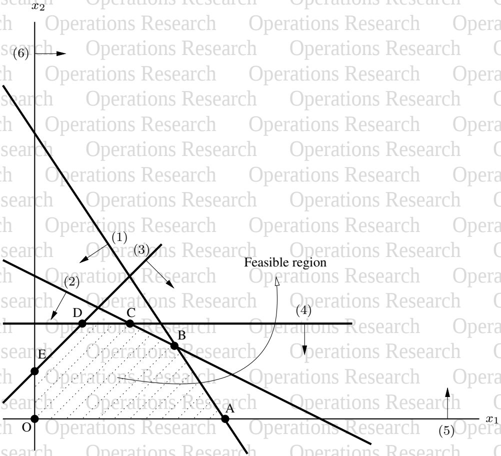

Figure: Feasible Region

- Each corner of the feasible region corresponds to the intersection of two of these lines: theoretically there are $\binom{6}{2} = 15$ such intersections.
- Only six of them are corners of the feasible region.
- In standard form this problem becomes (it has six variables)

$$\begin{array}{ll}
\text{minimize} & z = -5x_1 - 4x_2 \\
\text{subject to} & \\
 & \begin{aligned} 6x_1 + 4x_2 + s_1 &= 24 \\ x_1 + 2x_2 + s_2 &= 6 \\ -x_1 + x_2 + s_3 &= 1 \\ x_2 + s_4 &= 2 \\ x_1, x_2, s_1, s_2, s_3, s_4 &\geqslant 0 \end{aligned}
\end{array}$$

- The basis $\{x_2, s_2, s_3, s_4\}$ gives the basic (infeasible) solution

$$(x_1 \; x_2 \; s_1 \; s_2 \; s_3 \; s_4)^T = (0 \; 6 \; 0 \; -6 \; 1 \; 2)^T$$

which corresponds to an infeasible corner (intersection of (1) with (6)).

- The basis $\{x_1, x_2, s_1, s_3\}$ gives the basic feasible solution

$$(x_1 \; x_2 \; s_1 \; s_2 \; s_3 \; s_4)^T = (2 \; 2 \; 4 \; 0 \; 1 \; 0)^T$$

which corresponds to a feasible corner (intersection of (2) with (4)).

- The basis $\{x_1, x_2, s_2, s_4\}$ gives the basic infeasible solution

$$(x_1\ x_2\ s_1\ s_2\ s_3\ s_4)^T = (2\ 3\ 0\ -2\ 0\ -2)^T$$

which corresponds to an infeasible corner (intersection of (1) with (3)).

- The basis $\{x_1, x_2, s_3, s_4\}$ gives the optimal basic feasible solution

$$(x_1\ x_2\ s_1\ s_2\ s_3\ s_4)^T = (3\ 1.5\ 0\ 0\ 1.5\ 0.5)^T$$

which corresponds to an optimal feasible corner (intersection of (1) with (2)).

- If x is a basic feasible solution, once a set of basic variables has been selected, we can reorder the variables such that the basic variables are listed first:

$$x^T = (x_B^T \; x_N^T).$$

- The constraint matrix can be written (by rearranging the columns)

$$A = (B \; N),$$

where B has the columns corresponding to x_B, and those of N correspond to x_N.

- For a basic solution x we have $x_N = 0$ $(\in \mathbb{R}^{n-m})$, therefore the constraints $Ax = b$ become

$$A \cdot x = B \cdot x_B + N \cdot x_N = B \cdot x_B = b.$$

- The number of basic feasible solutions is upper bounded by the number of ways we can select m (basic) variables from the n existing variables:
$$\binom{n}{m} = \frac{n!}{m!(n-m)!}.$$
- This bound can be very large, but not all choices of basic variables may correspond to basic feasible solutions (see the above example).

In this section we will prove that the notion of basic feasible solution and that of extreme point coincide.

Theorem

Let $\mathcal{P} = \{\mathrm{x} \in \mathbb{R}^n : \mathrm{Ax} = \mathrm{b}, \mathrm{x} \geqslant 0\}$ *be a non-empty polyhedron and* $\mathrm{x} \in \mathcal{P}$. *Then, the following are equivalent*

(i) x *is an extreme point of* $\mathcal{P}$.

(ii) x *is a basic feasible solution.*

Proof. "$(i) \implies (ii)$" We prove by contradiction; first, we reorder the variables so that the non-zero variables come first: $\mathrm{x}^T = (\mathrm{x}_B^T\, \mathrm{x}_N^T)$, where $\mathrm{x}_N = 0$, $\mathrm{x}_B > 0$. Correspondingly, we can write $\mathrm{A} = (\mathrm{B}\ \mathrm{N})$ (but B may not be a square matrix). Obviously $\mathrm{B}\mathrm{x}_B = \mathrm{b}$.

If the columns of B are linearly independent, then the proof is done.

In what follows we suppose that the columns of B are not independent. Let B_j be the jth column of B; it must exist real numbers $y_1, \ldots, y_t$, not all of which are zero, such that $\sum_{j=1}^{t} y_j \mathrm{B}_j = 0$. If we put $\mathrm{y} = (y_1 \ \ldots \ y_t)^T$, then we have $\mathrm{By} = 0$. For every $\varepsilon \in \mathbb{R}_+^*$ we can write

$$B(\mathrm{x}_B \pm \varepsilon \mathrm{y}) = \mathrm{Bx}_b \pm \varepsilon \mathrm{By} = \mathrm{Bx}_B = \mathrm{b}. \tag{7}$$

Now, if ε is small enough, we must have $\mathrm{x}_B \pm \varepsilon \mathrm{y} > 0$; we define

$$\mathrm{x}^1 = \begin{pmatrix} \mathrm{x}_B + \varepsilon \mathrm{y} \\ \mathrm{x}_N \end{pmatrix}, x^2 = \begin{pmatrix} \mathrm{x}_B - \varepsilon \mathrm{y} \\ \mathrm{x}_N \end{pmatrix}.$$

From (7) it follows that $\mathrm{x}^{1,2} \in \mathcal{P}$; as $\mathrm{x}^1 \neq \mathrm{x}^2$ and $\mathrm{x} = 0.5\mathrm{x}^1 + 0.5\mathrm{x}^2$, x cannot be an extreme point - which is a contradiction.

"$(ii) \Longrightarrow (i)$" We may assume that the first variables are basic: $\mathrm{x}^T = (\mathrm{x}_B^T \ \mathrm{x}_N^T)$, where $\mathrm{x}_N = 0$, $\mathrm{x}_B \geqslant 0$, and $\mathrm{A} = (\mathrm{B} \ \mathrm{N})$ (B being an $m \times m$ matrix), with $\mathrm{B}\mathrm{x}_B = \mathrm{b}$, the columns of B being linearly independent, matrix B is non-singular.

The proof is also by contradiction: suppose that x is not an extreme point, then there exist two distinct points $\mathrm{x}^1, \mathrm{x}^2 \in \mathcal{P}$, and $\alpha \in (0,1)$, such that $\mathrm{x} = \alpha \mathrm{x}^1 + (1-\alpha)\mathrm{x}^2$. We write ${\mathrm{x}^1}^T = ({\mathrm{x}_B^1}^T \ {\mathrm{x}_N^1}^T)$ and ${\mathrm{x}^2}^T = ({\mathrm{x}_B^2}^T \ {\mathrm{x}_N^2}^T)$, with $\mathrm{x}_B^i, \mathrm{x}_N^i \geqslant 0, i = 1, 2$.

Obviously, we have $0 = \mathrm{x}_N = \alpha \mathrm{x}_N^1 + (1-\alpha)\mathrm{x}_N^2$, therefore, all terms being non-negative, $\mathrm{x}_N^1 = \mathrm{x}_N^2 = 0$. From here it follows that

$$\mathrm{B}\mathrm{x}_B = \mathrm{B}\mathrm{x}_B^1 = \mathrm{B}\mathrm{x}_B^2 = \mathrm{b} \Longrightarrow \mathrm{x}_B = \mathrm{x}_B^1 = \mathrm{x}_B^2 = \mathrm{B}^{-1}\mathrm{b},$$

which is again a contradiction. This completes the proof. □

A procedure for constructing basic solutions, for a linear program with a full-rank matrix is the following:

- Choose m linearly independent columns $A_{j_1}, A_{j_2}, \ldots, A_{j_m}$.
- Let $x_j = 0$ for all $j \in \{1, 2, \ldots, m\} \setminus \{j_1, j_2, \ldots, j_m\}$.
- Solve the system of m equations and m variables $Ax = b$.

If, in addition, the basic variables have non-negative values, then the basic solution will be feasible. Conversely, since every basic feasible solution is a basic solution, it can be obtained with this procedure.

- When one or more basic variables of a basic feasible solution are zero, the corresponding solution (or vertex) is called *degenerate*.
- At a degenerate vertex, several different bases correspond to the same basic feasible solution. *Degeneracy* can occur when the problem has a redundant constraint.
- Two extreme points are adjacent if they are connected by an "edge" of the feasible region. Two bases are *adjacent* if they share $(m-1)$ variables.
- Adjacent bases give *adjacent basic feasible solutions* - which may or may not be distinct.

- Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a convex set (say a polyhedron), $y \in \mathbb{R}^n \setminus \{0\}$ is a *direction of unboundedness* if

$$x + \alpha y \in \mathcal{M}, \forall\, x \in \mathcal{M}, \forall\, \alpha \in \mathbb{R}_+.$$

- Let $\mathcal{M} = \{x \in \mathbb{R}^n : Ax = b, x \geqslant 0\}$ (that is, a linear program in standard form) and $y \neq 0$ a direction of unboundedness for $\mathcal{M}$. It follows that $y \geqslant 0$ (exercise), and $x, (x + \alpha y)$ must be feasible points $(\forall\, \alpha \in \mathbb{R}_+)$ hence

$$Ax = b, A(x + \alpha y) = b \Longrightarrow Ay = 0.$$

- The reverse is also true: if $y \geqslant 0$, $y \neq 0$, and $Ay = 0$, then y is a direction of unboundedness.

Consider the linear program

$$\begin{array}{ll} \text{minimize} & z = -x_1 - 2x_2 \\ \text{subject to} & \\ & -2x_1 + x_2 \leqslant 2 \\ & -x_1 + x_2 \leqslant 3 \\ & x_1, x_2 \geqslant 0 \end{array}$$

- This problem has three extreme points $A(0,0)$, $A(0,2)$, and $B(1,4)$;
- Point $y = (1,0)^T$ is a direction of unboundedness, because $Ay = 0$ and $y \geqslant 0$.

Theorem

Consider the polyhedron $\mathcal{P} = \{\mathrm{x} \in \mathbb{R}^n_+ : \mathrm{Ax} = \mathrm{b}\}$ *representing the feasible region for problem (4), and* $\mathbb{E}_{\mathcal{P}} = \{\mathrm{v}^1, \mathrm{v}^2, \ldots, \mathrm{v}^t\}$. *If* $\mathcal{P}$ *is nonempty, then* $\mathbb{E}_{\mathcal{P}}$ *is nonempty, and every feasible solution* $\mathrm{x} \in \mathcal{P}$ *can be written as*

$$\mathrm{x} = \mathrm{y} + \sum_{i=1}^{t} \lambda_i \mathrm{v}^i, \tag{8}$$

where $\mathrm{Ay} = 0$ *(*y *is zero or is a direction of unboundedness), and* $\sum_{i=1}^{t} \lambda_i = 1, \lambda_i \geqslant 0, \forall i = \overline{1, i}$.

Proof. First we analyze the case when $\mathcal{P}$ is bounded: using corollary 2.1 we get that $\mathcal{P} = conv(\mathbb{E}_{\mathcal{P}})$, therefore (8) holds with $\mathrm{y} = 0$.
For the unbounded case, let x be a feasible solution of $\mathcal{P}$. We proceed by induction on the number of non-zero components of x. If x is a basic feasible solution, then it is an extreme point v^i of $\mathcal{P}$ and (8) holds with $\mathrm{y} = 0, \lambda_i = 1$, and $\lambda_j = 0$, for $j \neq i$.
If x is not an extreme point (i.e., it is not a basic feasible solution), the columns of A corresponding to the non-zero components of x are linearly dependent: there are n real numbers $y_1, y_2, \ldots, y_n$, not all of which are zero, such that

$$\sum_{i=1}^{n} y_i \mathrm{A}_i = 0 \text{ and } y_i = 0, \text{ if } x_i = 0.$$

Let $\mathrm{y} = (y_1, y_2, \ldots, y_n)^T$, we have $\mathrm{y} \neq 0$ and $\mathrm{Ay} = 0$. Hence $\mathrm{A}(\mathrm{x} + \varepsilon \mathrm{y}) = \mathrm{Ax} = \mathrm{b}$, that is, for small enough ε, $(\mathrm{x} + \varepsilon \mathrm{y}) \in \mathcal{P}$.

- Observe first that y and $-y$ cannot be both directions of unboundedness (why?).
- If y is not a direction of unboundedness, for some $\varepsilon > 0$, $(x + \varepsilon y)$ will meet the boundary of $\mathcal{P}$, then we can choose

$$\varepsilon_1 = \max\{\varepsilon > 0 \ : \ x + \varepsilon y \geqslant 0\}.$$

Obviously, $x^1 = (x + \varepsilon_1 y)$ has less non-zero components than x.

- In the same way, if $-y$ is not a direction of unboundedness, then we can find out an $\varepsilon_2 > 0$, such that $x^2 = (x - \varepsilon_2 y)$ has less non-zero components than x.

- If both y and $-y$ are not directions of unboundedness, then

$$x = \lambda x^1 + (1-\lambda)x^2, \text{ where } \lambda = \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2},$$

and we can apply the induction hypothesis for x^1 and x^2.

- If only y ($-y$) is a direction of unboundedness, then $x = y + x^1$ (respectively $x = y + x^2$), and, for x^i, we can apply the induction hypothesis.

□

Definition

An LP problem is called bounded *if it has a finite optimum, and* unbounded *when it has an infinite optimum.*

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