Operations Research - Lecture 1

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Some history

- Operations (or Operational) Research (OR) is an interdisciplinary branch of (applied) mathematics originated in UK during World War II;
- It started with the development of a radar defense system for the Royal Air Force;
- The term Operations Research is attributed to a RAF official after the initiation of teams to do *operational researches/analysis* on the communication system and the control of a radar station;
- Some believe that Charles Babbage (1791-1871) is the founding father of OR because he did researches on sorting and transportation of mail in England (which established the modern postal system - Penny Post in England).

- The new approach of picking an "operational" system and conducting "research" on how to make it run more efficiently soon started to expand into other arenas of the war.
- OR grew rapidly as many scientists realized that the principles that they had applied to solve problems for the military were equally applicable to many problems in the civilian sector.
- After the war, the ideas advanced in military operations were adapted to improve efficiency and productivity in the civilian sector.

Definitions

- *OR is a discipline that deals with the application of advanced analytic methods to help make better decisions. Employing techniques from other mathematical sciences, such as mathematical modeling, statistical analysis, and mathematical optimization, OR arrives at optimal or near-optimal solutions to complex decision-making problems.* (INFORMS site)
- *OR, application of scientific methods to the management and administration of organized military, governmental, commercial, and industrial processes.* (Encyclopædia Britannica)

To take advantage of the OR usefulness one must

- learn the *standard mathematical models* and *OR techniques*
- determine the solutions by using *algorithms*,
- understand innovative methods and practical issues related to the *use and development of computer implementations*.

Students need to know how to

- identify problems that the methods of OR can solve;
- structure the problems into standard mathematical models;
- apply or/and develop computational tools to solve the problems.

The aim of our OR course is to give the master student:

- a good foundation in the mathematics of OR and
- an appreciation of its potential applications.

The principal phases for implementing OR in practice include:

1. Definition of the problem.
2. Construction of the model.
3. Model solution.
4. Validation of the model.
5. Implementation of the solution.

1. Definition of the problem. Requires answering questions like:
   - What are the decision alternatives?
   - Under what restrictions is the decision made?
   - What is an appropriate objective function for evaluating the alternatives?

2. Construction of the model. The resulted model can fit one of the standard mathematical models or, if the model is too complex, it can be simplified using an heuristic approach, simulation, or a combination of these.

Phases of an OR Study

3. Model solution.

- This phase entails the use of well-defined optimization algorithms. A solution of the model is feasible if it satisfies all the constraints and is called optimal if, in addition of being feasible, it yields the best (maximum or minimum) value of the objective function.
- Another aspect of this phase is sensitivity analysis: obtaining additional information about the behavior of the optimum solution when the model undergoes some parameter changes.

4. Validation of the model. Model validity checks whether (or not) the proposed model does what it purports to do. As a common method: the model is valid if, under similar input conditions, it reasonably duplicates past performance. (We may use simulation as an independent tool for verifying the output of the mathematical model, if no historical data are available.)

5. Implementation of the solution. The translation of the results into understandable operating instructions to be issued to the people who will administer the recommended system.

The most prominent of the OR techniques is linear programming or LP. LP was designed for models with linear objective function and linear constraints; integer programming is a technique for LP problems in which some variables assume integer values.

- Linear Programming algorithms: Simplex/Dual Simplex.
- Integer Programming algorithms: Branch and Bound (B&B) algorithm, Cutting Plane algorithm.
- Interior Point algorithm is a general method for solving LP problems.

The simplex method was invented and developed by George Dantzig in 1947, based on his work for the U.S. Air Force. Even earlier, in 1939, L. V. Kantorovich (who was charged with the reorganization of the timber industry in the U.S.S.R.), formulated a restricted class of linear programs and a method for finding their solution.

Following *sciencing.com*:

- Food and Agriculture:
  - Farmers apply linear programming techniques to their work. By determining what crops they should grow, the quantity of it and how to use it efficiently, farmers can increase their revenue.
  - In nutrition, linear programming provides a powerful tool to aid in planning for dietary needs. In order to provide healthy, low-cost food baskets for needy families, nutritionists can use linear programming.
- Applications in Engineering:
  - Engineers also use linear programming to help solve design and manufacturing problems.
  - For example, in airfoil meshes, engineers seek aerodynamic shape optimization. This allows for the reduction of the drag coefficient of the airfoil.

- Transportation Optimization:
  - Transportation systems rely upon linear programming for cost and time efficiency.
  - Bus and train routes must factor in scheduling, travel time and passengers. Airlines use linear programming to optimize their profits according to different seat prices and customer demand. Airlines also use linear programming for pilot scheduling and routes.
- Efficient Manufacturing:
  - Manufacturing requires transforming raw materials into products that maximize company revenue. Each step of the manufacturing process must work efficiently to reach that goal.
  - For example, raw materials must past through various machines for set amounts of time in an assembly line.

Applications of LP

- Energy Industry:
  - Linear programming provides a method to optimize the electric power system design.
  - It allows for matching the electric load in the shortest total distance between generation of the electricity and its demand over time.
- *Delta Airlines* uses linear and integer programming in its Coldstart project to solve its fleet assignment problem. The problem is to match aircraft to flight legs and fill seats with paying passengers.
- *LibbeyOwens Ford* utilizes a large-scale linear programming model to achieve integrated production, distribution and inventory planning for its glass products. Schedulers and planners in the flat glass products group must coordinate production schedules for more than 200 different glass products.

- Some of the *general theoretical applications* of Linear/Integer Programming: the transportation model and its variants, the assignment model, the transshipment model, network models (The shortest-route problem, the maximal flow model, the critical path method (CPM)), the set-covering problem, the fixed-charge problem, the traveling salesperson (TSP) problem, capital budgeting.
- *Problems of interest to computer scientists* where linear/integer programming can be fruitful applied: maximum flow, rank aggregation, combinatorial (reverse) auctions, Markov decision processes, multi-agent systems, secret sharing schemes, linear time secure cryptography.

A manager of an oil refinery has 8 million barrels of crude oil A and 5 millions barrels of crude oil B allocated for production during the coming month. These resources can be used to make either gasoline, which sells for \$38 per barrel, or home heating oil, which sells for \$35 per barrel. There are three production processes with the following characteristics

	Process 1	Process 2	Process 3
input crude A	3	1	5
input crude B	5	1	3
output gasoline	4	1	3
output heating oil	3	1	4
Cost (\$)	51	11	40

All quantities are in barrels. Formulate a linear programming problem that would help the manager to maximise net revenue over the next month.

The corresponding LP (OR) model has three basic components:

1. the decision variables that we see to determine:
   - $x_1 =$ barrels of crude oil A used in the first process;
   - $x_2 =$ barrels of crude oil A used in the second process;
   - $x_3 =$ barrels of crude oil A used in the third process;

From these values the quantities of crude oil B used are easily computed:

- $\frac{5}{3}x_1 =$ barrels of crude oil B will be used in the first process;
- $\frac{1}{1}x_2 =$ barrels of crude oil B will be used in the second process;
- $\frac{3}{5}x_3 =$ barrels of crude oil B will be used in the third process;

Refinery Revenue - The Model

2. the objective (goal) that we need to optimize:
   - the revenue: $38 \cdot \left(\frac{4}{3}x_1 + \frac{1}{1}x_2 + \frac{3}{5}x_3\right) + 35 \cdot \left(\frac{3}{3}x_1 + \frac{1}{1}x_2 + \frac{4}{5}x_3\right) = \frac{257}{3}x_1 + 73x_2 + \frac{254}{5}x_3$.
   - the costs: $\frac{51}{3}x_1 + 11x_2 + \frac{40}{5}x_3 = 17x_1 + 11x_2 + 8x_3$
   - letting z represent the total profit (in \$), the objective will be
   - to maximize $z = \frac{206}{3}x_1 + 62x_2 + \frac{214}{5}x_3$.
3. the constraints that the solution must satisfy:
   - restrictions associated to current month availabilities of crude oil:
     $x_1 + x_2 + x_3 \leqslant 8,000,000$ (crude oil A);
     $\frac{5}{3}x_1 + x_2 + \frac{3}{5}x_3 \leqslant 5,000,000$ (crude oil B).
   - The non-negativity restrictions: $x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 0$.

The complete model becomes:

$$
\begin{aligned}
\text{maximize} \quad & z = \frac{206}{3}x_1 + 62x_2 + \frac{214}{5}x_3 \\
\text{subject to} \quad & \\
& x_1 + x_2 + x_3 \leqslant 8,000,000 \\
& \frac{5}{3}x_1 + x_2 + \frac{3}{5}x_3 \leqslant 5,000,000 \\
& x_1, x_2 \geqslant 0
\end{aligned}
$$

A company produces two kind of products. A product of the first type requires 1/4 hours of assembly labor, 1/8 hours of testing, and \$1.2 worth of raw materials. A product of the second type 1/3 hours of assembly, 1/3 hours of testing, and \$0.9 worth of raw materials. Given the existing work force, there can be at most 90 hours of assembly labor and 80 hours of testing, each day. Products of the first and second type have a market value of \$9 and \$8, respectively.

(i) Formulate a linear programming problem that can be used to maximize the daily profit of the company.

(ii) Consider the following two modifications to the original problem

(1) Suppose that up to 50 hours of overtime assembly labor can be scheduled, at a cost of \$7 per hour.

(2) Suppose that the raw material supplier provides a 10% discount if the daily bill is above \$300.

Which of the above two elements can be easily incorporated into the linear programming formulation and how? If one or both are not easily to incorporate, indicate how you might nevertheless solve the problem.

1. the decision variables of the model are:
   - $x_1 =$ quantity of the first product;
   - $x_2 =$ quantity of the second product.
2. the objective is to maximize the profit which is the difference between the revenues and costs:
   - revenues: $9x_1 + 8x_2$;
   - costs: $1.2x_1 + 0.9x_2$.
   - the objective is to maximize $z = (9x_1 + 8x_2) - (1.2x_1 + 0.9x_2) = 7.8x_1 + 7.1x_2$.

3. the constraints are:
   - associated with the assembly time: $\frac{1}{4}x_1 + \frac{1}{3}x_2 \leqslant 90$;
   - associated with the testing time: $\frac{1}{8}x_1 + \frac{1}{3}x_2 \leqslant 80$.
   - non-negativity restrictions: $x_1 \geqslant 0$, $x_2 \geqslant 0$.

The complete model will be:

$$
\begin{aligned}
\text{maximize} \quad & z = 7.8x_1 + 7.1x_2 \\
\text{subject to} \quad & \\
& 3x_1 + 4x_2 \leqslant 1,080 \\
& 3x_1 + 8x_2 \leqslant 1,920 \\
& x_1, x_2 \geqslant 0
\end{aligned}
$$

The first modification can be easily integrated in the linear model:

- the costs increase with \$350, therefore the objective becomes: maximize $z = 7.8x_1 + 7.1x_2 - 350$.
- the constraint associated with the assembly time becomes $\frac{1}{4}x_1 + \frac{1}{3}x_2 \leqslant 140$;

The complete model becomes:

$$\begin{aligned} \text{maximize} \quad & z = 7.8x_1 + 7.1x_2 - 350 \\ \text{subject to} \quad & \\ & 3x_1 + 4x_2 \leqslant 1{,}680 \\ & 3x_1 + 8x_2 \leqslant 1{,}920 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

Considerations for the second modification of the original problem:

- if the bill for the raw material exceeds \$300, i.e., if $1.2x_1 + 0.9x_2 \geqslant 300$, then the costs with raw material will be $0.9(1.2x_1 + 0.9x_2) = 1.08x_1 + 0.81x_2$;
- a possible[1] solution may be to solve the original problem, and, if the raw material bill exceeds \$ 300 for the optimal solution, solve the problem with a new objective: maximize $z = 7.92x_1 + 7.19x_2$.

The complete model becomes:

$$
\begin{aligned}
\text{maximize} \quad & z = 7.92x_1 + 7.19x_2 \\
\text{subject to} \quad & \\
& 3x_1 + 4x_2 \leqslant 1{,}080 \\
& 3x_1 + 8x_2 \leqslant 1{,}920 \\
& x_1, x_2 \geqslant 0
\end{aligned}
$$

[1] Is this an optimal solving of the modified problem?

Ozarks Farms uses at least 800 lb of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions:

	lb per lb of feed-stuff		
Feed-stuff	Protein	Fiber	Cost ($/lb)
Corn	0.09	0.02	0.30
Soybean meal	0.60	0.06	0.90

The dietary requirements of the special feed are at least 30% protein and at most 5% fiber. Ozark Farms wishes to determine the daily minimum cost feed mix.

1. the decision variables of the model are:
   - x_1 = lb of corn in the daily mix;
   - x_2 = lb of soybean meal in the daily mix.
2. the objective:
   - letting z represent the cost for lb of special feed, the objective of the company is
   - to minimize $z = 0.3x_1 + 0.9x_2$.

3. the constraints are:
   - the dietary requirements: $x_1 + x_2 \geqslant 800$;
   - the protein dietary requirement: $0.09x_1 + 0.6x_2 \geqslant 0.3(x_1 + x_2)$;
   - the fiber dietary requirement: $0.02x_1 + 0.06x_2 \leqslant 0.05(x_1 + x_2)$.
   - non-negativity restrictions: $x_1 \geqslant 0$, $x_2 \geqslant 0$.

The complete model is:

$$\begin{aligned} \text{minimize} \quad & z = 0.3x_1 + 0.9x_2 \\ \text{subject to} \quad & \\ & x_1 + x_2 \geqslant 800 \\ & 0.21x_1 - 0.3x_2 \leqslant 0 \\ & 0.03x_1 - 0.01x_2 \geqslant 0 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

Reddy Mikks Company ([Taha07]) produces both interior and exterior paints from two raw materials, M_1 and M_2

	Tons of raw material/ton of		Maximum daily
	exterior paint	interior paint	availability (tons)
Raw material M_1	6	4	24
Raw material M_2	1	2	6
Profit per ton ($1000)	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than one ton. Also, the maximum daily demand for interior paint is 2 tons. Reddy Mikks wants to determine the optimal (best) product mix of interior and exterior paints that maximizes the total daily profit.

The correspondingly LP (OR) model has three basic components:

1. the decision variables that we see to determine:
   - x_1 = tons produced daily of exterior paint;
   - x_2 = tons produced daily of interior paint.
2. the objective (goal) that we need to optimize:
   - if z represent the total daily profit, the objective of the company is
   - to maximize $z = 5x_1 + 4x_2$[2].

[2] Thousands of \$.

3. the constraints that the solution must satisfy:
   - restrictions associated to daily availabilities of M_1 and M_2:
     $6x_1 + 4x_2 \leqslant 24$ (Raw material M_1);
     $x_1 + 2x_2 \leqslant 6$ (Raw material M_2).
   - market restrictions:
     $x_2 - x_1 \leqslant 1$ (market limit);
     $x_2 \leqslant 2$ (demand limit).
   - non-negativity restrictions:
     $x_1 \geqslant 0$, $x_2 \geqslant 0$.

The complete linear model:

$$
\begin{aligned}
\text{maximize} \quad & z = 5x_1 + 4x_2 \\
\text{subject to} \quad & \\
& 6x_1 + 4x_2 \leqslant 24 \\
& x_1 + 2x_2 \leqslant 6 \\
& x_2 - x_1 \leqslant 1 \\
& x_2 \leqslant 2 \\
& x_1, x_2 \geqslant 0
\end{aligned}
$$

- Linear programs can be solved geometrically or algebraically, both approaches being equivalent.
- Geometrically solving is based on the geometry of the possible solutions set (feasible region) - this set is a convex one in a certain euclidean space.
- One of the advantages of this approach is that the form of the constraints does not influence the process of solving; on the other hand the algebraic approaches is heavily based on specific form of the constraints.
- Using geometry, many of the central concepts in linear programming become easier to understand.
- The only clear disadvantage is that the geometric method successfully applies only in *two dimensions*.

Consider the following linear program

$$\begin{aligned} \text{maximize} \quad & z = x_1 + 2x_2 \\ \text{subject to} \quad & \\ & 2x_1 + x_2 \leqslant 12 \\ & x_1 + x_2 \geqslant 5 \\ & -x_1 + 3x_2 \geqslant 3 \\ & 6x_1 - x_2 \geqslant 12 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

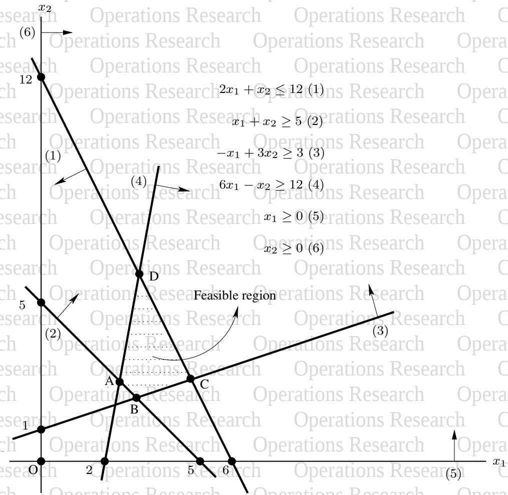

Figure: Feasible Region

Graphical Solution - Optimal Solution

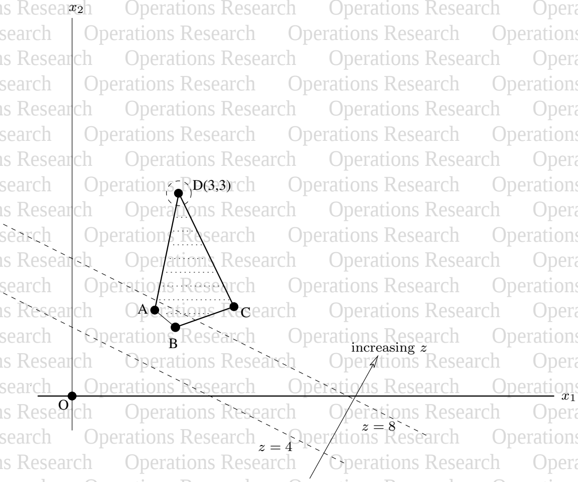

Figure: Optimal Solution

Consider now one of the previous examples (Diet Problem):

$$\begin{aligned} \text{minimize} \quad & z = 0.3x_1 + 0.9x_2 \\ \text{subject to} \quad & \\ & x_1 + x_2 \geqslant 800 \\ & 0.21x_1 - 0.3x_2 \leqslant 0 \\ & 0.03x_1 - 0.01x_2 \geqslant 0 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

Or, equivalently,

$$\begin{aligned} \text{minimize} \quad & z = 3x_1 + 9x_2 \\ \text{subject to} \quad & \\ & x_1 + x_2 \geqslant 800 \\ & 7x_1 - 10x_2 \leqslant 0 \\ & 3x_1 - x_2 \geqslant 0 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

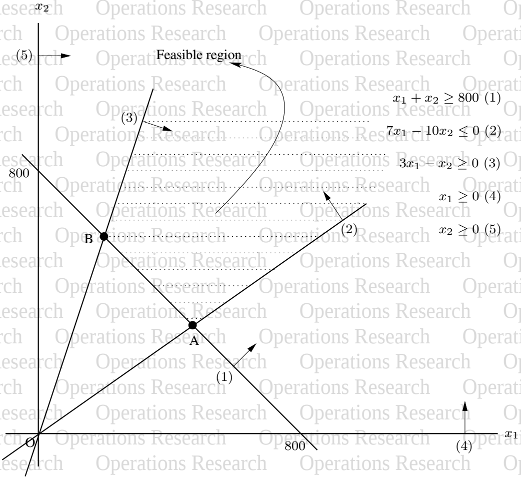

Figure: Feasible Region

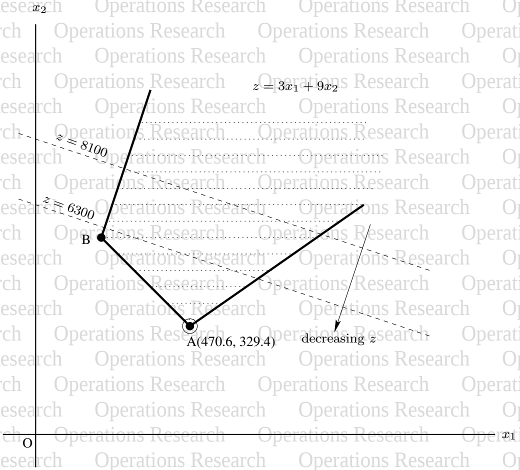

Figure: Optimal Solution

Consider now the Reddy Mikks example:

$$\begin{aligned} \text{maximize} \quad & z = 5x_1 + 4x_2 \\ \text{subject to} \quad & \\ & 6x_1 + 4x_2 \leqslant 24 \\ & x_1 + 2x_2 \leqslant 6 \\ & x_2 - x_1 \leqslant 1 \\ & x_2 \leqslant 2 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

Graphical Solution - Feasible Region

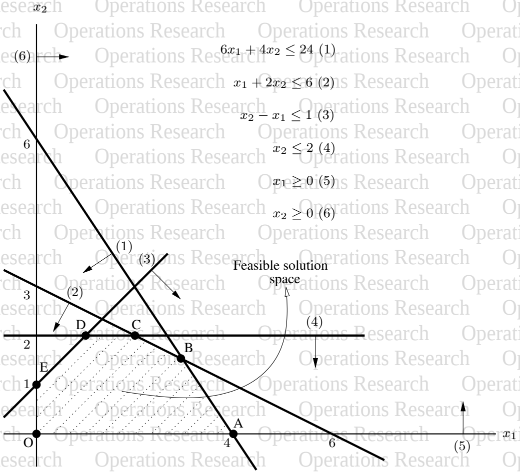

Figure: Feasible Region

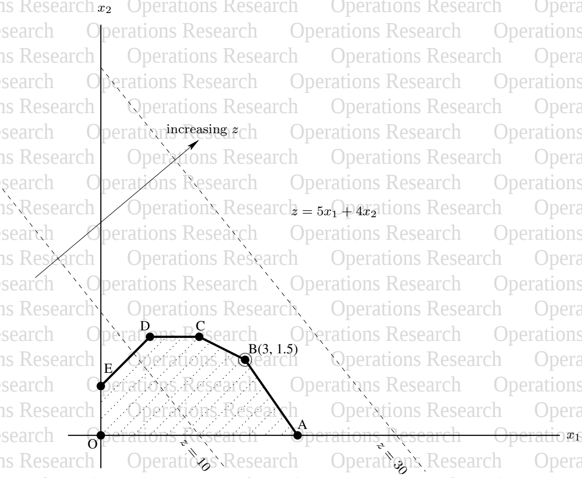

Figure: Optimal Solution

The graphical procedure includes two steps:

1. Determination of the feasible region or feasible solution space.
   - the non-negativity of the variables restricts the solution-space area to the *first quadrant* (the so-called positive orthant);
   - for the remaining constraints:
     - first replace each inequality with an equation and then graph the resulting line;
     - next, consider the effect of the inequality: the line divides the plane into two *half-planes* or *semi-planes* and only one of these two halves satisfies the inequality (we use an arrow to point to the feasible semi-plane);
     - the intersection of all these semi-planes gives the feasible region.

2. Finding of the optimal solution from among all the feasible points in the solution space.

- first, identify the *direction* in which the profit function *improves* (increases for maximizing z, or decreases for minimizing z);
- we can do so by assigning two arbitrary values to z, which is equivalent to graphing two lines;
- the optimal solution occurs at a *corner* (a point in the feasible region) beyond which any further improvement will put z outside the feasible region.

- An important characteristic of the optimum LP solution (if any) is that it is always associated with a *corner* or *extreme point* of the feasible region (where two or more lines intersect).
- This is true even if the objective function happens to be parallel to a constraint-line (in which case it is possible that any point on that line segment will be an alternative optimum, but the important observation here is that the line segment is totally defined by its corner points).

- We will prove that the geometric approach is equivalent to the algebraic one. In order to do this we will describe first particular (*standard* and *canonical*) forms of the constraints.
- Standard forms will be used to define a *basic feasible solution*; the algebraic notion of a basic feasible solution is equivalent to the geometric notion of an extreme point.
- This is of great value because, in higher dimensions, basic feasible solution are easier to generate than extreme points. In this way we can see why the algebraic method is more practical than the geometric one.

Towards an Algebraic Solution

- It will be shown that any feasible solution (or feasible point) can be represented in terms of basic feasible solutions (extreme points). This leads to show that any linear program with a finite optimal solution has an optimal extreme point.
- This last result will greatly motivate the introduction of the *simplex algorithm*: a method that solves a linear program by examining basic feasible solutions (that is, extreme points) one by one, until an optimal one is found.

- A *matrix* of dimensions $m \times n$ is an array of numbers a_{ij}:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Its (i,j)-th entry is a_{ij} or $[A]_{ij}$. A_j it is the jth *column* of matrix A, and a'_i is its ith *row*:

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix} = \begin{bmatrix} a'_1 \\ a'_2 \\ \vdots \\ a'_m \end{bmatrix}$$

- The *transpose* of a $m \times n$matrix A is the following $n \times m$ matrix

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

- A vector $x \in \mathbb{R}^n$ is a column and has components $x_1, x_2, \ldots x_n$.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ its transpose is } x^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}.$$

The euclidean norm of x is $||x||_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$.

- When x is a vector, $x \geqslant 0$ ($x > 0$) means that every component of x is non-negative (positive). Similar meaning have the notations $A \geqslant 0$, $A > 0$, for a matrix A.
- The *inner product* of two vectors $x, y \in \mathbb{R}^n$ is

$$x^T y = y^T x = \langle x, y \rangle = \sum_{i=0}^{n} x_i \cdot y_i$$

- If A is a $m \times n$ matrix and $x \in \mathbb{R}^n$, then

$$Ax = \sum_{i=1}^{n} x_i A_i = \begin{bmatrix} \langle a'_1, x \rangle \\ \langle a'_2, x \rangle \\ \vdots \\ \langle a'_m, x \rangle \end{bmatrix}$$

Algebra background

- The vectors $x^1, x^2, \ldots x^p \in \mathbb{R}^n$ are called *linearly independent* if, for every $\alpha_1, \alpha_2, \ldots, \alpha_p \in \mathbb{R}$, we have

$$\alpha_1 x^1 + \alpha_2 x^2 + \cdots \alpha_p x^p = 0 \Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_p = 0.$$

  Otherwise $x^1, x^2, \ldots x^p \in \mathbb{R}^n$ are called *linearly dependent*.
- If $x^1, x^2, \ldots x^p \in \mathbb{R}^n$ are linearly independent, then $p \leqslant n$. $\mathbb{R}^n$ has n linearly independent vectors - which a *base* in $\mathbb{R}^n$.
- If $A \in \mathbb{R}^{m \times n}$, the *rank* of A, *rank*(A), is the maximum number of linearly independent rows of A (which equals the maximum number of linearly independent columns of A).
- Thus, $rank(A) \leqslant \max\{m, n\}$. A is said to have *full row rank* if $rank(A) = m$ and A have *full column rank* if $rank(A) = n$.

- Consider a set of boolean variables $X = \{x_1, x_2, \ldots, x_n\}$, and a *conjuctive normal form (CNF)* formula, like, for e. g.,

$$F = (x_3 \vee \overline{x}_4) \wedge (x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee \overline{x}_3 \vee x_4).$$

- The *SAT problem* is to find an assignment of truth over X such that a given CNF formula is true.
- The integer LP model for the SAT problem (e. g., [Wang01]):
  - each boolean variable becomes a LP variable x_i, and each negative literal $\overline{x}_i$ is transformed in $1 - x_i$;
  - the logical operator or ($\vee$) is replaced by addition operator ($+$);
  - each disjunction of literals (clause) is replaced by a an inequality of the following type: the sum of the LP variables of the corresponding boolean variables is greater or equal with 1.

- For the CNF formula F from above:

$$
\begin{array}{lcl}
x_3 \vee \overline{x}_4 & \text{becomes} & x_3 + (1 - x_4) \geqslant 1 \\
x_1 \vee \overline{x}_2 \vee x_3 & \text{becomes} & x_1 + (1 - x_2) + x_3 \geqslant 1 \\
\overline{x}_1 \vee \overline{x}_3 \vee x_4 & \text{becomes} & (1 - x_1) + (1 - x_3) + x_4 \geqslant 1
\end{array}
$$

- To solve this SAT instance is to find a solution to the following set of inequalities:

$$
\begin{aligned}
x_3 - x_4 &\geqslant 0 \\
x_1 - x_2 + x_3 &\geqslant 0 \\
-x_1 - x_3 + x_4 &\geqslant -1 \\
x_i \in \{0, 1\}, i &= \overline{1, 3}
\end{aligned}
$$

- An LP problem can be written like follows:

$$\begin{aligned} \min\ z = y_1 + y_2 + y_3 \\ x_3 - x_4 + y_1 &\geqslant 0 \\ x_1 - x_2 + x_3 + y_2 &\geqslant 0 \\ -x_1 - x_3 + x_4 + y_3 &\geqslant -1 \\ \mathrm{x} \in \{0,1\}^4, \mathrm{y} \in \{0,1\}^3 \end{aligned}$$

- More general, to a SAT instance corresponds an ILP problem

$$\begin{aligned} \min\ z = 1^T \mathrm{y} \\ A\mathrm{x} + I_m \mathrm{y} &\geqslant 1 - \mathrm{b} \\ \mathrm{x} \in \{0,1\}^n, \mathrm{y} \in \{0,1\}^m \end{aligned}$$

where A is the coefficient $m \times n$ matrix of x, I_m is the $m \times m$ identity matrix, and b is the vector containing the number of negative literals in each clause.

- This is one of the simplest of all integer LP problems.
- The problem is to select a maximum value collection of n objects subject to restriction on some consumed resources (like weight or volume).
- Suppose that item j has weight b_j and value c_j; we add a boolean variable for each item j: $x_j = 1$ iff item j is included in our collection.
- The problem becomes:

$$\max z = \mathbf{c}^T \mathbf{x}$$
$$\mathbf{b}^T \mathbf{x} \leqslant b$$
$$\mathbf{x} \in \{0,1\}^n$$

where b is the maximum weight of the knapsack.

- Among the possible variants: replace $\mathbf{x} \in \{0,1\}^n$ with $\mathbf{x} \in \mathbb{Z}_+^n$ and you will get the *unbounded knapsack problem*.

- We have a graph $G = (V, E)$ and we want to *color its vertices* such that adjacent vertices have different colors - using as few colors as possible.
- The integer LP model for the vertex coloring problem (e. g., [Wang01])
  - to each possible color (no more than $n = |V|$), j, we associate a boolean variable: $w_j = 1$ iff color j is used by some vertex;
  - to each pair (vertex, color), (u, j), we introduce a boolean variable: $x_{uj} = 1$ iff the vertex u is colored with j;
  - to each edge $uv \in E$ and each color j we add an inequality which constrains at most one of the end points to receive the color j: $x_{uj} + x_{vj} \leqslant w_j$.
  - each vertex must be in the end colored: exactly one of the variables $(x_{uj})_{1 \leqslant j \leqslant n}$ must be 1.

- Hence the integer LP model for coloring the vertices of a given graph becomes ([Mendez09])

$$
\begin{aligned}
\min z = 1^T \mathrm{w} & \\
\sum_{j=1}^{n} x_{uj} &= 1, \quad \forall u \in V \\
x_{uj} + x_{vj} &\leqslant w_j, \quad \forall uv \in E, 1 \leqslant j \leqslant n \\
x_{uj}, w_j \in \{0,1\} &
\end{aligned}
$$

- Sometimes the contraints $w_{j+1} \leqslant w_j$, $j = \overline{1,n}$ are added in order to inforce that color $j+1$ not to be used if color j is not used.

- Consider a salesman traveling from city to city (from a given list). The salesman starts in a city and has to visit all cities on a business trip before returning home. The problem then consists of *finding the shortest tour which visits every city once*.
- TSP can be modelled using a simple (and undirected) weighted graph $G = (V, E)$, $w : E \to \mathbb{R}_+$ being a distance or a cost function defined on the edges.
- The integer LP model for TSP:
  - to each edge, uv, we associate a boolean variable: $x_{uv} = 1$ the tour uses edge uv - note that $x_{uv} = x_{vu}$, this is the symmetrical version of the problem;
  - since the traveler must enter and leave each city (vertex) exactly once, we must have

$$\sum_{v \in V} x_{uv} = 2$$

- However the last restriction doesn't eliminate the subtours (instead of just one tour we can get a list with disjoint subtours which covers all edges).
- In order to have only one tour each subgraph induced by sets travelled edges must be a forest except when all vertices (cities) belong to that subgraph.
- Therefore the integer LP model for TSP is

$$
\begin{aligned}
\min z = \sum_{uv \in E} w_{uv} x_{uv} & & & \\
\sum_{v \in V} x_{uv} &= 2, & & \forall u \in V \\
\sum_{u,v \in U, u \neq v} x_{uv} &\leqslant |U| - 1, & & \forall \emptyset \neq U \subsetneq V \\
x_{uv} \in \{0, 1\} & & &
\end{aligned}
$$

- A variant of TSP: using a directed graph (i. e., x_{uv} doesn't necessary equal x_{vu}).

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