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- We already discussed some examples of discrete optimization problems that can be modeled as *ILP* problems.
- A **perfect formulation** of a set $\mathcal{M} \subseteq \mathbb{R}^n$ is a linear system of inequalities $Ax \leq b$ such that $\text{conv}(\mathcal{M}) = \{x \in \mathbb{R}^n : Ax \leq b\}$, i. e., $\text{conv}(\mathcal{M})$ is a polyhedra.
- When such a formulation is available for a mixed integer linear set, the corresponding integer mixed linear program can be solved as a linear program.
- A classical example, already studied, is when the constraint matrix is *totally unimodular*. Important combinatorial problems on directed or bipartite graphs have a totally unimodular constraint matrix.

- First, we will consider again the subject of totally unimodular matrices.
- Second, we will study a new situation when a perfect formulation can be obtained: *totally dual integral systems*.

Definition

A **polyhedron** \mathcal{P} is **integral** if, for every $c \in \mathbb{R}^n$, for which $\sup\{c^T x : x \in \mathcal{P}\} \in \mathbb{R}$, the supremum is attained by an integral vector.

Proposition

A rational polyhedron ($\mathcal{P} = \{x : Ax \leq b\}$, $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$) is integral iff every extreme point of it is an integral vector.

Hoffman and Kruskal Result

- Another way to introduce the totally unimodular matrices is to ask the following question "Which integral matrices have the property that the polyhedron $\{x \in \mathbb{R}^n : Ax \leq b\}$ is integral for every integral vector b ?"
- The answer is given in the following result of Hoffman and Kruskal.

Theorem

Let A be an integral $m \times n$ matrix. Then A is totally unimodular iff, for each $b \in \mathbb{Z}^m$, the polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n : Ax \leq b\}$ is integral.

Definition

A full row rank matrix A is called **unimodular** if it is integral and each square non-singular sub-matrix with the same rank as A has determinant ± 1 .

The following result is left as an exercise:

Proposition

An $m \times n$ matrix A is totally unimodular if and only if the matrix $[I_m \ A]$ is unimodular.

The proof of the Hoffman and Kruskal result is based on an auxiliary result due to Veinott and Dantzig.

Theorem

Let A be an integral $m \times n$ matrix with $\text{rank}(A) = m$. Then the polyhedron $\mathcal{P} = \{x \in \mathbb{R}_+^n : Ax = b\}$ is integral for each vector $b \in \mathbb{Z}^m$ if and only if A is unimodular.

Proof. " \Leftarrow " Let $b \in \mathbb{Z}^m$ and x be an extreme point of \mathcal{P} . Then the columns of A corresponding to nonzero components of x are linearly independent. We can add some other columns in order to obtain m linearly independent columns of A . Let B the sub-matrix formed with these columns and let $x' \in \mathbb{R}^m$ be the restriction of x to the columns of B . We must have $\det(B) = \pm 1$, therefore $x' = B^{-1}b$ is integral and x is also integral.

" \implies " Let B be a $m \times m$ sub-matrix of A with the same rank as A . $\det(B) = \pm 1$ if and only if $B^{-1}u$ is integral for each integral vector u (why?). We will check this property: let $u \in \mathbb{Z}^m$ and v an integral vector such that $w = v + B^{-1}u \geq 0$.

Let $b = Bw = Bv + u$; b must be integral; we can add zero components to w to obtain an \mathbb{R}^n vector \bar{w} . Hence, we have $A\bar{w} = b$, and \bar{w} must be an extreme point of \mathcal{P} (why?). Then \bar{w} , w , and $B^{-1}u$ must be integral vectors. \square

Proof (of theorem 2.1) A is totally unimodular iff the matrix $[I_m \ A]$ is unimodular. Moreover, for each $b \in \mathbb{Z}^m$, the extreme points of $\{x \in \mathbb{R}_+^n : Ax \leq b\}$ are integral iff the extreme points of $\{w \in \mathbb{R}_+^{m+n} : [I_m \ A]w = b\}$ are integral. \square

Corollary

Let A be an integral $m \times n$ matrix. Then the polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n : d_1 \leq x \leq d_2, b_1 \leq Ax \leq b_2\}$ is integral for all integral vectors b_1, b_2, d_1, d_2 if and only if A is totally unimodular.

Proof. Observe first that A is totally unimodular iff the following matrix is also totally unimodular

$$\tilde{A} = \begin{bmatrix} I_n \\ -I_n \\ A \\ -A \end{bmatrix}$$

Furthermore $\mathcal{P} = \{z \in \mathbb{R}^n : \tilde{A}z \leq \tilde{b}\}$, where $\tilde{b}^T = (d_2^T, -d_1^T, b_2^T, -b_1^T)$.

□

Corollary

An integral $m \times n$ matrix, A , is totally unimodular iff each (and every) of the following polyhedra is integral $\{x : x \leq d, Ax \leq b\}$, $\{x : x \leq d, Ax \geq b\}$, $\{x : x \geq d, Ax \leq b\}$, $\{x : x \geq d, Ax \geq b\}$.

Corollary

An integral $m \times n$ matrix A is totally unimodular iff for all integral vectors b and c , the duality equation

$$\max \{c^T x : x \geq 0, Ax \leq b\} = \min \{b^T y : y \geq 0, A^T y \geq c\}$$

is achieved by integral vectors x and y (provided that both problems are bounded).

- Related to integral polyhedra, J. Edmonds and R. Giles proved the following interesting result.

Theorem

Consider the LP problem $\max \{c^T x : Ax \leq b\}$, where A, b, c are rational. If, for every integral c , the maximum is integer whenever is finite, then the maximum is attained by an integral vector x , for each c , whenever this maximum is finite.

- Hence, if b is integral and the dual problem

$$\min \{b^T y : y \geq 0, A^T y = c\}$$

has an integral optimal solution for each integral c , then the primal problem has also an integral optimal solution.

- These considerations motivate the following concept.

Definition

A rational system $Ax \leq b$ is **totally dual integral** (or **TDI**) if, for every integral $c \in \mathbb{Z}^n$, for which $\sup\{c^T x : Ax \leq b\} \in \mathbb{R}$, its dual $\min\{b^T y : y \geq 0, A^T y = c\}$ has an integral optimal solution.

- The importance of this type of linear systems is given by theorem of J. Edmonds and R. Giles, which can be restated like follows.

Theorem

Let $Ax \leq b$ be a totally dual integral system with b an integral vector. Then $\mathcal{P} = \{x : Ax \leq b\}$ is an integral polyhedron.

There are integral polyhedra $\{x : Ax \leq b\}$ with integral data for which the system $Ax \leq b$ is not TDI. However any integral polyhedron can be represented by a TDI system with integral data.

Theorem

Let \mathcal{P} be a rational polyhedron. There exists a TDI system $Ax \leq b$ with an integral matrix A , such that $\mathcal{P} = \{x : Ax \leq b\}$. If \mathcal{P} is an integral polyhedron, then the vector b can be chosen to be integral.

Total Dual Integrality

Proof. Let $\mathcal{P} \neq \emptyset$ a rational polyhedron, $\mathcal{P} = \{x : Bx \leq d\}$, where $B \in \mathbb{Z}^{m \times n}$ and $d \in \mathbb{Z}^m$. Define

$$\mathcal{C} = \{y \in \mathbb{R}^n : y = B^T u, 0 \leq u \leq 1\}, \mathcal{C}_i = \mathbb{Z}^n \cap \mathcal{C}.$$

Since \mathcal{C} is bounded, \mathcal{C}_i is finite. For $x \in \mathcal{P}$ and $c \in \mathcal{C}$ we have

$$c^T x = u^T Bx \leq u^T d, \text{ for a certain } 0 \leq u \leq 1. \quad (1)$$

Therefore, we can define, for every $c \in \mathcal{C}_i$, $\delta_c = \max \{c^T x : x \in \mathcal{P}\}$. We define the system $Ax \leq b$ as the family of all inequalities $c^T x \leq \delta_c$, with $c \in \mathcal{C}_i$.

Obviously, $\mathcal{P} \subseteq \{x : Ax \leq b\}$; let $i \in \{1, 2, \dots, m\}$ and define u such that $u_i = 1$ and $u_j = 0$ for $j \neq i$; $c = B^T u$ ($c \in \mathcal{C}_i$) is the i th row of B and equation (1) reads $c^T x \leq d_i$, for all $x \in \mathcal{P}$. Therefore every inequality of the system $Bx \leq d$ makes part of the system $Ax \leq b$, and $\{x : Ax \leq b\} \subseteq \mathcal{P}$.

Total Dual Integrality

We will show next that the system $Ax \leq b$ is TDI. Let $c \in \mathbb{Z}^n$ such that $\max\{c^T x : Ax \leq b\} = \alpha \in \mathbb{R}$. Obviously, $\alpha = \max\{c^T x : x \in \mathcal{P}\}$. Let y^* an optimal solution to the dual of the latter problem $\min\{d^T y : B^T y = c, y \geq 0\} = \alpha = d^T y^*$. Define $u = y^* - \lfloor y^* \rfloor$ ($0 \leq u \leq 1$), $c_1 = B^T u$, $c_2 = B^T \lfloor y^* \rfloor$ ($c = c_1 + c_2$); obviously, u is an optimal solution to $\min\{d^T y : B^T y = c_1, y \geq 0\}$ and $\lfloor y^* \rfloor$ is an optimal solution to $\min\{d^T y : B^T y = c_2, y \geq 0\}$. $c_1 = c - c_2 \in \mathbb{Z}^n$, hence $c_1 \in C_i$. We can suppose that the system $Bx \leq d$ is composed from the first m inequalities of $Ax \leq b$ - which has $k \geq m$ rows. Assume that $c_1^T x \leq \delta_{c_1}$ is the h th inequality of $Ax \leq b$.

The h th column of the identity matrix in $\mathbb{R}^{k \times k}$, e_k^h , is an optimal solution to the problem $\min \{b^T z : A^T z = c_1, z \geq 0\}$. On the other hand, the vector \bar{z} defined by $\bar{z}_i = \lfloor y_i^* \rfloor$, if $i = \overline{1, m}$, and $\bar{z}_i = 0$, for $i = \overline{m+1, k}$ is an optimal solution to the problem $\min \{b^T z : A^T z = c_2, z \geq 0\}$. Therefore $z^* = \bar{z} + e_k^h$ is an integral optimal solution to the problem $\min \{b^T z : A^T z = c, z \geq 0\}$. \square

- We introduce in this subsection an integral polyhedron defined by a TDI system of inequalities.

Definition

Let S be a nonempty finite set. A set function $\varphi : 2^S \rightarrow \mathbb{R}$ is called **submodular** if $\varphi(A) + \varphi(B) \geq \varphi(A \cup B) + \varphi(A \cap B)$, for all $A, B \subseteq S$.

Definition

Let $N = \{1, 2, \dots, n\}$ and $\varphi : 2^N \rightarrow \mathbb{R}$ be submodular function. The **submodular polyhedron** associated with φ is $\mathcal{P}(\varphi) = \{x \in \mathbb{R}^n : \sum_{j \in J} x_j \leq \varphi(J), \forall J \subseteq N\}$.

Theorem

Let $N = \{1, 2, \dots, n\}$ and $\varphi : 2^N \rightarrow \mathbb{Z}$ be submodular function with $\varphi(\emptyset) = 0$. Then the system $\sum_{j \in J} x_j \leq \varphi(J)$, $J \subseteq N$ is TDI; subsequently the submodular polyhedron $\mathcal{P}(\varphi)$ is integral.

Proof. Consider the pair of primal dual problems

$$(P) \max \left\{ c^T x : \sum_{j \in J} x_j \leq \varphi(J), J \subseteq N \right\} \text{ and}$$

$$(D) \min \left\{ \sum_{J \subseteq N} \varphi(J) y_J : \sum_{J \ni j} y_J = c_j, j \in N, y \geq 0 \right\}.$$

Let $c \in \mathbb{Z}^n$ such that (P) is bounded (which is equivalent with $c \geq 0$ - why?), and assume that $c_1 \geq c_2 \geq \dots \geq c_n \geq 0$. Let $J_0 = \emptyset$, $J_i = \{1, 2, \dots, i\}$, $i = \overline{1, n}$.

Define $x_i^* = \varphi(J_i) - \varphi(J_{i-1})$, $i = \overline{1, n}$; x^* is integral and we will show that it is an optimal solution to (P) . Let $J \subseteq N$, we show that $\sum_{j \in J} x_j^* \leq \varphi(J)$,

by induction on $|J|$. If $J = \emptyset$, then we have nothing to prove. Suppose that $\sum_{i \in I} x_i^* \leq \varphi(I)$, for all $I \subseteq N$, $|I| < t$. Let $I \subseteq N$, $|I| = t$, and

$i_0 = \max I$; we have $I \subseteq J_{i_0}$, and, by induction hypothesis, $\sum_{i \in I \setminus \{i_0\}} x_i^* \leq \varphi(I \setminus \{i_0\})$. Now,

$$\begin{aligned} \sum_{i \in I} x_i^* &= \sum_{i \in I \setminus \{i_0\}} x_i^* + x_{i_0}^* \leq \varphi(I \setminus \{i_0\}) + x_{i_0}^* \\ &= \varphi(I \setminus \{i_0\}) + \varphi(J_{i_0}) - \varphi(J_{i_0-1}) \leq \varphi(I) \text{ - by submodularity of } \varphi. \end{aligned}$$

We form now a feasible integral solution, y^* , for (D) whose objective value is $c^T x^*$. Define

$$y_J^* = \begin{cases} c_i - c_{i+1}, & \text{if } J = J_i, i = 1, n-1 \\ c_n, & \text{if } J = N \\ 0, & \text{otherwise.} \end{cases}$$

y^* is feasible solution for the dual since $y^* \geq 0$ and

$$\sum_{J \supseteq j} y_J^* = \sum_{h=j}^n y_{J_h}^* = (c_j - c_{j+1}) + \dots + (c_{n-1} - c_n) + c_n = c_j, \forall j \in N.$$

On the other hand

$$\begin{aligned} \sum_{J \subseteq N} \varphi(J) y_J^* &= \sum_{i=1}^{n-1} \varphi(J_i) (c_i - c_{i+1}) + \varphi(N) c_n = \\ \sum_{i=1}^n [\varphi(J_i) - \varphi(J_{i-1})] c_i &= \sum_{i=1}^n x_i^* c_i = c^T x^*. \end{aligned}$$

Hence, by strong duality theorem, x^* and y^* are optimal (and integral) solutions for their problems. \square

We can observe that the above demonstration give an algorithm (*a greedy algorithm*) to compute an optimal solution of (P) : we order the variables so that $c_1 \geq c_2 \geq \dots \geq c_n \geq 0$, and define $x_i^* = \varphi(J_i) - \varphi(J_{i-1})$, $i = \overline{1, n}$, where $J_i = \{1, 2, \dots, i\}$, $i = \overline{1, n}$, and $J_0 = \emptyset$.

The Matching Polytope

- We give here an example of perfect formulation for a classical problem and discuss its TDIness.
- Let $G = (V, E)$ be an undirected graph, and \mathcal{M}_G be the family of its matchings: a set $M \subseteq E$ is a matching if $\{u, v\} \cap \{x, y\} = \emptyset$, for every $uv, xy \in M$.
- Let $x \in \{0, 1\}^E$ be the characteristic vector of a matching $M \in \mathcal{M}_G$ ($x = x^M$). Obviously, this vector satisfies the following inequalities:

$$\sum_{e \ni v} x_e \leq 1, \forall v \in V. \quad (2)$$

$$x_e \geq 0, \forall e \in E. \quad (3)$$

Definition

The matching polytope^a of a graph G is the convex hull of the characteristic vectors for all matchings in \mathcal{M}_G :

$$\mathcal{P}_{\text{matching}}(G) = \text{conv} \left(\{x^M : M \in \mathcal{M}_G\} \right).$$

^aA polytope is a bounded polyhedron.

- The question we can address regarding this polytope is: the set of inequalities (2) - (3) is a perfect formulation of $\mathcal{P}_{\text{matching}}(G)$ (that is, these inequalities characterize any point in the matching polytope)?

The Matching Polytope

- In general, the answer is no, as we can see from the following example: let G be K_3 , and take $x_e = 0.5$, for every $e \in E(K_3)$; x satisfies (2) - (3), but does not belong to the matching polytope.
- J. Edmonds ([Edmonds65]) proved that it is enough to add the following set of inequalities in order to get a set of inequalities that fully determines the matching polytope:

$$\sum_{e \in E[U]} x_e \leq \left\lfloor \frac{|U|}{2} \right\rfloor, \forall U \subseteq V, |U| \equiv 1 \pmod{2}. \quad (4)$$

Theorem

The matching polytope of a graph G is fully determined by the set of inequalities (2) - (4).

The Matching Polytope

- We consider now the problem of finding a maximum weight matching in a given graph.
- Suppose that we have $w : E \rightarrow \mathbb{R}$; we define the following pair of primal/dual problems:

$$(P) \left\{ \begin{array}{l} \text{maximize} \quad w^T x, \\ \text{s. t.} \quad \sum_{e \ni v} x_e \leq 1, \forall v \in V \\ \sum_{e \in E[U]} x_e \leq \left\lfloor \frac{|U|}{2} \right\rfloor, \forall U \subseteq V, |U| \equiv 1 \pmod{2} \\ x \in \mathbb{R}_+^E. \end{array} \right.$$

The Matching Polytope

$$(D) \begin{cases} \text{minimize} & \sum_{v \in V} y_v + \sum_{U \in \mathcal{P}_{\text{odd}}(V)} z_U \left\lfloor \frac{|U|}{2} \right\rfloor, \\ \text{s. t.} & y_u + y_v + \sum_{U \in \mathcal{P}_{\text{odd}}(V), u, v \in U} z_U \geq w_{uv}, \forall uv \in E \\ & \mathbf{z} \in \mathbb{R}_+^{\mathcal{P}_{\text{odd}}(V)}, \mathbf{y} \in \mathbb{R}_+^V. \end{cases}$$

- Let us denote by $\nu(w)$ the maximum weight of a matching in G with respect to w . Cunningham and Marsh ([Cunningham72]) proved

Theorem

The matching polytope of a graph G is TDI. Furthermore, for every integral w , there exists two integral vectors y and z , forming an optimal solution to (D), such that the family $\{U \subseteq V : z_U > 0\}$ is laminar.

Definition

Let C be a nonempty set. A family $\mathcal{F} \subseteq 2^C$ is called *laminar* if, for every $A, B \in \mathcal{F}$, we have $A \subseteq B$, $B \subseteq A$, or $A \cap B = \emptyset$.

Proof of theorem 3.6. Let $w : E \rightarrow \mathbb{Z}$; we can suppose, without loss of the generality, that $w : E \rightarrow \mathbb{Z}_+$, as we can delete the edges having a negative weight. We use induction on $|E| + \sum_{e \in E} w(e)$ to show that problem (D) has an integral solution.

Case 1: there exists a vertex v_0 which is saturated by any maximum-weight matching in G . Define a new weight $w' : E \rightarrow \mathbb{Z}_+$

$$w'(e) = \begin{cases} w(e) - 1, & \text{if } e = uv_0, \\ w(e), & \text{otherwise.} \end{cases}$$

Obviously, $\nu(w') = \nu(w) - 1$ (why?). By inductive hypothesis, there exists y', z' , an integral optimal solution with respect to w' such that $\{U \subseteq V : z'_U > 0\}$ is laminar. Let us define $z^* = z'$ and y^* like follows

$$y_v^* = \begin{cases} y'_{v_0} + 1, & \text{if } v = v_0, \\ y'_v, & \text{otherwise.} \end{cases}$$

It is easy to see that y^*, z^* is a dual feasible (integral) solution with respect to w , having the objective value $\nu(w') + 1 = \nu(w)$ (which is optimal).

Case 2: no vertex is saturated by every maximum-weight matching.

Let y, z be a fractional optimal dual solution; we must have $y = 0$ (for, if $y_v > 0$, by complementary slackness, $\sum_{e \ni v} x_e = 1$, hence v must be covered by all maximum-weight matchings).

We choose this optimal solution in such a way that maximizes

$$\sum_{U \in \mathcal{P}_{\text{odd}}(V)} z_U \left\lfloor \frac{|U|}{2} \right\rfloor^2.$$

We will show that \mathbf{z} is integral and $\{U \subseteq V : z_U > 0\}$ is laminar.

Suppose that $\mathcal{F}_w = \{U \subseteq V : z_U > 0\}$ is not laminar, that is, there exist $U, W \in \mathcal{F}_w$ such that $U \cap W \neq \emptyset$. We prove first that $U \cap W$ is odd. Let $v \in U \cap W$, and $M \in \mathcal{M}_G$ a maximum-weight matching which misses v . Since $z_U > 0$, by complementary slackness

$$\sum_{e \in E[U]} x_e = \left\lfloor \frac{|U|}{2} \right\rfloor,$$

that means that M contains exactly $\left\lfloor \frac{|U|}{2} \right\rfloor$ edges inside U ; therefore v is the only vertex missed by M inside U . In a similar manner it can be shown that v is the only vertex missed by M inside W .

The Matching Polytope

Therefore any vertex from $U \cap W \setminus \{v\}$ is covered in M by edges having both endpoints in $U \cap W \setminus \{v\}$, hence $|U \cap W \setminus \{v\}|$ is even (i. e., $|U \cap W|$ is odd).

Let $\epsilon < z_U, z_W$, form a new vector z' by adding ϵ to $z_{U \cap W}, z_{U \cup W}$, and subtracting ϵ from z_U, z_W . We prove that z' is dual feasible and optimal.

Obviously, the sums $\sum_{T \in \mathcal{P}_{\text{odd}}(V), u'v' \in E[T]} z'_T$ does not change for $u', v' \in$

$U \cap W$ or $u', v' \in U \setminus W$ or $u', v' \in W \setminus U$ - with $u'v' \in E$ (why?).

On the other hand the objective value does not change (why?). So z' is

still a dual feasible optimal solution. However $\sum_{U \in \mathcal{P}_{\text{odd}}(V)} z'_U \left[\frac{|U|}{2} \right]^2$ it is

increased by $\epsilon |U \setminus W| \cdot |W \setminus U| / 2$ (check!) - which is a contradiction.

Hence $\mathcal{F}_w = \{U \subseteq V : z_U > 0\}$ must be a laminar family.

The Matching Polytope

Suppose now that z is not integral, and choose $U \in \mathcal{F}_w$ a maximal set such that z_U is fractional. Let $U_1, U_2, \dots, U_h \in \mathcal{F}_w$ be all maximal subsets of U (these sets must be mutually disjoint); denote $\alpha = z_U - \lfloor z_U \rfloor$, and form a new vector z' by adding α to each z_{U_i} and subtracting α from z_U . z' is still dual feasible; for proving this we need to consider the edge constraints in the dual - namely for edges from $E[U]$.

Let $uv \in E$ with $u, v \in U$. If $u, v \in U_i$







$$\sum_{T \in P_{\text{odd}}(V), uv \in E[T]} z'_T = \sum_{T \in P_{\text{odd}}(V), uv \in E[T]} z_T \text{ (why?).}$$

If $uv \in e[U] \setminus e[U_i]$, for every $i = \overline{1, h}$, then U is the only set from $P_{\text{odd}}(V) \cap \mathcal{F}_w$ which contains uv . The dual constraint for uv reads $z_U \geq w(uv)$, thus $z'_U \geq w(uv)$.

The Matching Polytope

Since U_i are disjoint odd subsets of U , we have $\left\lfloor \frac{|U|}{2} \right\rfloor > \sum_{i=1}^h \left\lfloor \frac{|U_i|}{2} \right\rfloor$ (check!).

This contradicts the optimality of z and the proof ends. \square

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