Game Theory - Lecture 10

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- Game theory is a formal, mathematical discipline which studies situations of competition and cooperation between several involved parties.
- Its applications range from strategic questions in warfare to understanding economic competition, from economic and social problems of fair distribution to behavior of animals in competitive situations, from parlor games to political voting systems - and this list is certainly not exhaustive.

- "Game-theoretic" situations can be recognized in the Bible or the Talmud;
  - in the work (over 2000 years old) of the Chinese warrior-philosopher Sun Tzu;
  - in the works of A. Cournot and J. Bertrand on price competition;
  - an early application of zero-sum games to the political problem of parliamentary representation in some of the work of C.L. Dodgson (better known as Lewis Carroll, the writer of *Allice's Adventures in Wonderland*).
- More formal works on game theory: Zermelo, John von Neumann (1928).
- John von Neumann and Oskar Morgenstern, Theory of Games and Economic Behavior, 1944.
- John Nash, Harsanyi, Selten, Shapley etc.

Areas of overlap between CS and game theory ([Halpern03], [Qiu03]):

- Complexity theory
  - making sense of resource-bounded reasoning,
  - sanity check on algorithms, e.g. for auctions.
- Distributed computing
  - Much the same problems as game theory: many agents, uncertainty, possibly different goals;
  - Nevertheless, difference in emphasis: fault tolerance/scalability versus strategic concern.
- Artificial Intelligence
  - concerned with knowledge representation, reasoning, planning, learning. Example: on-line auctions.
- Cryptography: interaction between mutually distrustful parties.

Branches of Game Theory:

- Non-cooperative game theory:
  - Games in normal form ('one-shot' games): zero-sum games and nonzero-sum games
  - Extensive form games
- Cooperative game theory:
  - Transferable Utility (TU) games
  - Non-Transferable Utility (NTU) games
- Bargaining games.

- The usual distinction between *cooperative* and *non-cooperative game theory* is that in a cooperative game binding agreements between players are possible, whereas this is not the case in non-cooperative games.
- A more workable distinction between cooperative and non-cooperative games can be based on the "modelling technique" that is used: in a non-cooperative game players have explicit *strategies*, whereas in a cooperative game players and coalitions are characterized, more abstractly, by the *outcomes* and *payoffs* that they can reach.

Story. (South-Pacific in 1943) The Japanese Admiral Imamura has to transport troops across the Bismarck Sea to New Guinea, and the American Admiral Kenney wants to bomb the transport. Imamura has two possible choices: the Northern route (2 days) or the Southern route (3 days). Kenney must choose one of these routes to send his planes too. If he chooses the wrong route he can call back the planes and send them to the other route, but the number of bombing days is reduced by 1. We assume that the number of bombing days represents the payoff to Kenney in a positive sense and to Imamura in a negative sense.

Model. The Battle of the Bismarck Sea problem can be modeled as:

		Imamura	
		North	South
Kenney	North	2	2
	South	1	3

Kenney (player 1) chooses a row; Imamura (player 2) chooses a column. This game is an example of a zero-sum game because the sum of the payoffs is always equal to zero.

Solution.

- This game is easy to analyze because one of the players has a *weakly dominant choice*, i.e. a choice which is always at least as good (giving always at least as high a payoff) as any other choice, no matter what the opponent decides to do.
- By choosing North, Imamura is always at least as well off as by choosing South.
- So, it is safe to assume that Imamura chooses North, and Kenney, being able to perform this same kind of reasoning, will then also choose North, since that is the best reply to the choice of North by Imamura.

- Another way to look at this game is to observe that the payoff of 2 resulting from the combination (North, North) is maximal in its column ($2 \geqslant 1$) and minimal in its row ($2 \geqslant 2$).
- Such a position in the matrix is called a saddle point. In such a saddle point, neither player has an incentive to deviate unilaterally (the strategy profile (North, North) is a Nash equilibrium).
- Also observe that, in such a saddle point, the row player maximizes his minimal payoff (because $2 = \min\{2, 2\} \geqslant 1 = \min\{1, 3\}$), and the column player (who has to pay according to our convention) minimizes the maximal amount that he has to pay (because $2 = \max\{2, 1\} \leqslant 3 = \max\{2, 3\}$).
- The resulting payoff of 2 from player 2 to player 1 is called the value of the game.

Comments. Two-person zero-sum games with finitely many choices, like the one above, are also called matrix games since they can be represented by a single matrix. The combination (North, North) in the example above corresponds to what happened in reality back in 1943 (see the memoirs of Winston Churchill).

Story. In the game of matching pennies, both players have a coin and simultaneously show heads or tails. If the coins match, player 2 gives his coin to player 1; otherwise, player 1 gives his coin to player 2.

Model. This is a zero-sum game with payoff matrix

	Heads	Tails
Heads	1	-1
Tails	-1	1

Solution.

- Observe that there is no saddle point.
- Thus, there does not seem to be a natural way to solve the game. Von Neumann proposed to solve games like this - and zero-sum games in general - by allowing the players to *randomize between their choices*.
- Here, suppose that player 1 chooses heads or tails both with probability 0.5. Suppose furthermore that player 2 plays heads with probability q and tails with probability $(1-q)$, where $0 \leqslant q \leqslant 1$.
- In this case the expected payoff for player 1 is (independent of q)

$$\frac{1}{2} \cdot [q \cdot 1 + (1-q) \cdot (-1)] + \frac{1}{2} \cdot [q \cdot (-1) + (1-q) \cdot 1] = 0.$$

- So by randomizing in this way between his two choices, player 1 can guarantee to obtain 0 in expectation (of course, the actually realized outcome is always $+1$ or -1).
- Analogously, player 2 by playing heads or tails each with probability 0.5, can guarantee to pay 0 in expectation. Thus, the amount of 0 plays a role similar to that of a saddle point.
- Again, we will say that 0 is the value of this game.

Comments. The randomized choices of the players are usually called mixed strategies. Randomized choices are often interpreted as *beliefs* of the other player(s) about the choice of the player under consideration. Von Neumann proved that every two-person game has a value if the players can use mixed strategies: this is the minimax theorem.

Story.

- Two prisoners (players 1 and 2) have committed a crime together and are interrogated separately.
- Each prisoner has two possible choices: he/she may 'cooperate' (C) which means 'not betray his/her partner' or he/she may 'defect' (D), which means 'betray his/her partner'.
- The punishment for the crime is 10 years of prison. Betrayal yields a reduction of 1 year for the traitor - if both "betray his/her partner" - and freedom otherwise. If a prisoner is not betrayed, he is convicted to 1 year for a minor offense.

Model. This situation can be summarized as follows:

	C	D
C	$(-1,-1)$	$(-10,0)$
D	$(0,-10)$	$(-9,-9)$

There are two payoffs at each position: by convention the first number is the payoff for player 1 and the second number is the payoff for player 2. Observe that the game is no longer zero-sum, and we have to write both numbers at each matrix position.

Comments.

- For both players D is a strictly dominant choice: for each player, D is (strictly) the best choice, whatever the other player does.
- So, it is natural to argue that the outcome of this game will be the pair of choices (D, D), leading to the payoffs $(-9, -9)$.
- Thus, due to the existence of strictly dominant choices, the Prisoners' Dilemma game is easy to analyze.

The Prisoners' Dilemma is a metaphor for many economic (an outstanding example is the so-called tragedy of the commons) and CS situations (such as in ISP (Internet Service Provider) routing context).

Story. Consider two ISPs each having its own separate network. The two networks can exchange traffic via two transit points, called peering points, C and S (as shown in Figure 1). We also have two origin-destination pairs s_i and t_i each crossing between the domains, with t_i very close to S.

- Suppose that ISP 1 needs to send traffic from point s_1 in his own domain to point t_1 in 2nd ISP's domain. ISP 1 has two choices for sending its traffic, corresponding to the two peering points.
- ISPs typically behave selfishly and try to minimize their own costs, and send traffic to the closest peering point, as the ISP with the destination node must route the traffic, no matter where it enters its domain.

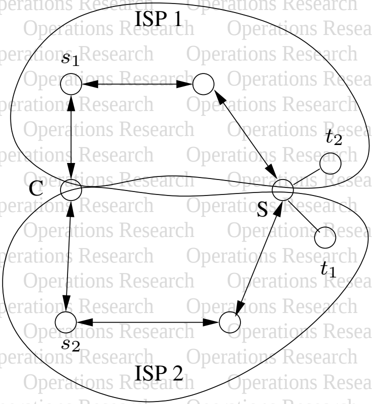

Figure: The ISP routing problem.

- Since C is closer, using this peering point ISP 1 incurs a cost of 1 unit (in sending traffic along one edge), whereas if it uses the farther peering point S, it occurs a cost of 2.
- The farther peering point S is more directly on route to the destination t_1: routing through S results in shorter overall path.
- The length of the path through C is 4 while through S is 2, as the destination is very close to S. Assume that symmetrically the ISP 2 needs to send traffic from s_2 to t_2.
- Each ISP has two choices, one is better from a selfish perspective (route through peering point C), but hurts the other player. The two choices of the two ISPs lead to a game with the matrix cost:

	C	S
C	$(4,4)$	$(1,5)$
S	$(5,1)$	$(2,2)$

- Consider a game with the following *payoff matrix*

$$\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mn} \end{pmatrix}.$$

- The rows are *pure strategies* for player A, and the columns the are pure strategies for player B.
- The payoffs are of player A: if A chooses strategy i and B chooses j, then the player A wins p_{ij} and B loses p_{ij}.

- A strategy i of A is *weakly dominated* by a strategy l of A if

$$p_{lj} \geqslant p_{ij}, \forall j.$$

- A strategy j of B is *weakly dominated* by a strategy h of B if

$$p_{ih} \leqslant p_{ij}, \forall i.$$

- We eliminate sequentially all dominated strategies (for A and B). If we get only one strategy for each player, i and j, then these are the strategies of our players and the value of the game is p_{ij}.
- Consider the following non-cooperative zero-sum game

Strategy		player B		
		1	2	3
player A	1	1	1	3
	2	1	3	2
	3	1	0	1

- For A: strategy 3 is dominated by strategy 1 , and we eliminate it.
- For B: strategies 2, 3 are dominated by strategy 1: we eliminate it.
- For A: strategy 1 is dominated by strategy 2, and we eliminate it.

		B		
		1	2	3
A	1	1	1	3
	2	1	3	2
	3	1	0	1

→

		B		
		1	2	3
A	1	1	1	3
	2	1	3	2

→

		B
		1
A	1	1
	2	1

→

		B
		1
A	2	1

- Player A choose strategy 2, and B strategy 1, game value is 1.

- When we cannot find dominated strategies we apply the *maximin criterion*:
- Player A chooses a strategy which maximizes the minimum payoff among all its remaining strategies

$$1 \leqslant l \leqslant n \text{ a. î. } \underline{v} = \max_i \min_j p_{ij} = \min_j p_{lj}.$$

- Player B chooses a strategy which minimizes the maximum loss among all its remaining strategies

$$1 \leqslant h \leqslant m \text{ a. î. } \overline{v} = \min_j \max_i p_{ij} = \max_i p_{ih}.$$

- If the two positions coincide we have a *saddle point*: this pair of strategies is optimal for both players, neither player can deduce an advantage from this *stable solution*. The value of the game is $v = \underline{v} = \overline{v}$.

- Consider the following non-cooperative zero-sum game

Strategy		player B		
		1	2	3
player A	1	-2	-1	5
	2	3	1	2
	3	4	-1	-2

- There are no dominated strategies.
- For A (maximin): $\underline{v} = \max\{-2, 1, -2\} = 1 = p_{22}$.
- For B (minimax): $\overline{v} = \min\{4, 1, 5\} = 1 = p_{22}$.
- p_{22} is a *saddle point*, the game value is 1. The corresponding strategies are $(2, 2)$.

Solving Using Mixed Strategies (by Linear Programming)

- Denote by:
  - $(x_1, x_2, \ldots, x_m)$ the mixed strategy of player 1;
  - $(y_1, y_2, \ldots, y_n)$ the mixed strategy of player 2;
- The strategy $(x_1, x_2, \ldots, x_m)$ is optimal if the expected payoff for player A is at least the value of the game, v. That is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_i y_j \geqslant v,$$

for all mixed (and pure) strategies of player B. This is equivalent to

$$\sum_{i=1}^{m} p_{ij} x_i \geqslant v, \forall\, 1 \leqslant j \leqslant n.$$

- Additional (linear) constraints are

$$x_1 + x_2 + \ldots + x_m = 1,$$

$$x_i \geqslant 0, \forall\, 1 \leqslant i \leqslant m.$$

- We replace the unknown constant v by the variable x_{m+1}
- Player A will find his optimal mixed strategies by using the Simplex method to solve the following LP problem:

$$\begin{cases} \max z = x_{m+1} \\ \sum_{i=1}^{m} p_{ij} x_i - x_{m+1} \geqslant 0, j = \overline{1, n} \\ \sum_{i=1}^{m} x_i = 1 \\ x_i \geqslant 0, i = \overline{1, m} \\ x_{m+1} \in \mathbb{R} \end{cases}$$

- By proceeding in a way which is completely analogous to that just described, player B would conclude that his optimal strategy is given by an optimal solution to the following LP problem:

$$\begin{cases} \min w = y_{n+1} \\ \sum_{j=1}^{n} p_{ij} y_j - y_{n+1} \leqslant 0, i = \overline{1, m} \\ \sum_{j=1}^{n} y_j = 1 \\ y_j \geqslant 0, j = \overline{1, n} \\ y_{n+1} \in \mathbb{R} \end{cases}$$

- The LP problem for player B and the LP problem for player A are dual to each other. This fact has several important implications:
- The optimal mixed strategies for both players can be found by solving only one of the LPs because the optimal dual solution is an automatic by-product of the simplex method calculations to find the optimal primal solution.
- This brings all *duality theory* to bear upon the interpretation and analysis of matrix games. It provides a simple proof of the minimax theorem.

Story. Cities 1, 2, and 3 want to be connected with a nearby power source. The possible transmission links and their costs are shown in figure 2. If the cities cooperate in hiring the link they save on the hiring costs (the links have unlimited capacities).

Model. The players in this situation are the three cities. Denote the player set by $N = \{1, 2, 3\}$. These players can form coalitions: any subset S of N is called a *coalition*. Denote by $c(S)$ the cost for the cheapest route connecting the cities in the coalition S with the power source. The cost-savings $v(S)$ for coalition S are equal to the difference in costs corresponding to the situation where all members of S work together and the situation where all members of S work alone. The pair (N, v) is called a *cooperative* (reward) *game*.

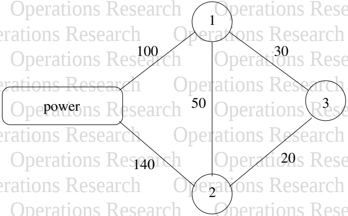

Figure: Situation leading to the three cities game.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$c(S)$	100	140	130	150	130	150	150
$v(S)$	0	0	0	90	100	120	220

Solution.

- Basic questions:
  - which coalitions will actually be formed?
  - how should the proceedings (savings) of such a coalition be distributed among its members?
- To form a coalition the consent of every member is needed, but it is likely that the willingness of a player to participate in a coalition depends on what the player obtains in that coalition.
- It is usually assumed that the 'grand coalition' N of all players is formed, and the question is then reduced to the problem of distributing the amount $v(N)$ among the players. In other words, we look for vectors $x = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ such that $x_1 + x_2 + x_3 = 220$, where player i obtains x_i.

- Equal share does not really reflect the asymmetry in the situation: some coalitions save more than others. The literature offers many quite different solutions to this distribution problem, among which are the *core*, the *Shapley value*, and the *nucleolus*.
- The core consists of all vectors (x_1, x_2, x_3) satisfying also $x(S) \geqslant v(S)$ for all S. For this example, the core is large, the Shapley value is $(65, 75, 80)$, and the nucleolus is $(56\,2/3, 76\,2/3, 86\,2/3)$.

Comments. The implicit assumptions in a game like this are, first, that a coalition can make binding agreements on the distribution of its payoff and, second, that any payoff distribution that distributes (or, at least, does not exceed) the savings or, more generally, worth of the coalition is possible. For these reasons, such games are called *cooperative games with transferable payoff* or *utility* (TU games).

Definition

A cooperative game *consists of*

- *a set of players - N (which is the grand coalition);*
- *for each coalition $S \subseteq N$, a set of actions a_S;*
- *for each player, preferences over the set of all actions of all coalitions of which she is a member.*

- An outcome of a cooperative game consists of a partition of the set of players into groups, together with an action for each group in the partition.
- The player's preferences are specified by giving a payoff function that represents them.

Definition

A cooperative game with transferable utility *is a cooperative game in which the set of payoff distributions resulting from each coalition's actions may be represented by a single number* $v(S)$*, equal to the total output it can obtain (total "payoff" that may be distributed among the members of the coalition)* $v : 2^N \to \mathbb{R}$ *the payoff function also called the* *characteristic function.*

- The total payoff of any coalition S in a game with transferable payoff is the *worth* of S.
- Usually, the payoff function has the following two properties:

Cooperative Games with Transferable Utility

- *superadditivity:* $v(S \cup T) \geqslant v(S) + v(T)$, for every two disjoint coalitions $S, T \subseteq N$, $S \cap T = \varnothing$. The value of a union of disjoint coalitions is no less than the sum of the coalitions' separate values.
- *monotonicity:* $v(S) \leqslant v(T)$, if $S \subseteq T$. Larger coalitions gain more.

Definition

A game with transferable payoff $G = \langle N, v \rangle$ is cohesive if

$$v(N) \geqslant \sum_{S \in \Sigma} v(S), \forall\, \Sigma \text{ a partition of } N$$

(The whole is greater than the sum of its parts.)

Clearly, a superadditive game is cohesive.

- *Story.* A landowner's estate, when used by k workers, produces the output $f(k)$ of food, where f is a (strictly) increasing function for which $f(0) = 0$. The total number of workers is m. The landowner and each worker care only about the amount of output she receives, and prefer more to less.
- *Players.* $N = \{1, 2, \ldots, m + 1\}$, where 1 is the landowner and $2, \ldots, m + 1$ are the workers.
- *Actions.* A coalition consisting solely of workers has a single action in which no member receives any output. The set of actions of a coalition S consisting of the landowner and k workers is the set of all S-allocations of the output $f(k + 1)$ among the members of S.
- This game is cohesive because the grand coalition produces more output than any other coalition, and, for any partition of the set of all the players, only one coalition produces any output.

Definition

- *A payoff vector* $\mathrm{x} = (x_1, x_2, \ldots, x_n)$ *is a vector of proposed amounts to be received by the players, with the understanding that player* i *is to receive* x_i.
- *A payoff vector* x *is called allocation if is efficient (or group rational), i.e.,*

$$\sum_{i=1}^{n} x_i = v(N)$$

- *An allocation* x *is called imputation if is individually rational, that is*

$$x_i \geqslant v(\{i\}), \forall 1 \leqslant i \leqslant n.$$

For a cohesive game, the set of imputations is never empty: since

$$\sum_{i=1}^{n} v(\{i\}) \leqslant v(N),$$

$x_i = v(\{i\})$, for $i = \overline{1, n-1}$, and $x_n = v(N) - \sum_{i=1}^{n-1} v(\{i\})$ defines an imputation. It's easy to see that the set of imputations is exactly the simplex consisting of the convex hull of the following n points

$$\left\{ \mathrm{x} \in \mathbb{R}^n \ : x_j = v(\{j\}), \forall j \neq i, x_i = v(N) - \sum_{j \neq i} x_j \right\}$$

Example: consider a three-person game having the payoffs ($v(\varnothing) = 0$):

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	.5	0	.75	3	2.5	2	9

The set of imputations is

$$\left\{\mathrm{x} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 9, x_1 \geqslant .5, x_2 \geqslant 0, x_3 \geqslant .75\right\}$$

- A game is called *essential* if $\sum_{i=1}^{n} v(\{i\}) < v(N)$, otherwise is *inessential*. If a game is inessential there is only one imputation $\mathrm{x} = (v(\{1\}), v(\{2\}), \ldots, v(\{n\}))$.
- A superadditive inessential game is trivial, as, for every coalition S, $v(S) = \sum_{i \in S} v(\{i\})$ (players have no interest to form coalitions).

Definition

An imputation x *is said to be unstable through a coalition* S *if* $v(S) > \sum_{i \in S} x_i$. *If* x *is an imputation unstable through at least one coalition, then* x *is unstable. Otherwise,* x *is stable. The set,* $\mathcal{C}$, *of all stable imputations is called the core of the game:*

$$\mathcal{C} = \left\{ \mathrm{x} \in \mathbb{R}^n \ : \ \sum_{i=1}^{n} x_i = v(N), v(S) \leqslant \sum_{i \in S} x_i, \forall S \subseteq N \right\}.$$

- The core may contain many points, but the core can also be empty.
- One may take the size of the core as a measure of stability, or of how likely it is that a negotiated agreement is prone to be set.

Example: consider a three-person game having the payoffs ($v(\varnothing) = 0$):

S	{1}	{2}	{3}	{1, 2}	{1, 3}	{2, 3}	{1, 2, 3}
$v(S)$	1	0	1	4	3	5	8

The set of imputations is

$$\left\{\mathrm{x} \in \mathbb{R}^3 \ : x_1 + x_2 + x_3 = 8, x_1 \geqslant 1, x_2 \geqslant 0, x_3 \geqslant 1\right\}$$

Which imputations are unstable?

$v(\{1,2\}) = 4 \Rightarrow (x_1, x_2, x_3)$ with $x_1 + x_2 < 4$ are unstable through $\{1,2\}$.

$v(\{1,3\}) = 3 \Rightarrow (x_1, x_2, x_3)$ with $x_1 + x_3 < 3$ are unstable through $\{1,3\}$.

$v(\{2,3\}) = 5 \Rightarrow (x_1, x_2, x_3)$ with $x_2 + x_3 < 5$ are unstable through $\{2,3\}$.

The core is the remaining set of points:

$$\mathcal{C} = \left\{\mathrm{x} \in \mathbb{R}^3 \ : x_1 + x_2 + x_3 = 8, x_1 \geqslant 1, x_2 \geqslant 0, x_3 \geqslant 1, \right.$$

$$\left. x_1 + x_2 \geqslant 4, x_1 + x_3 \geqslant 3, x_2 + x_3 \geqslant 5\right\}$$

- The concept of the core is useful as a measure of stability. As a solution concept, it presents a set of imputations without distinguishing one point of the set as preferable to another. Indeed, the core may be empty.
- This section deals with the concept of a value. In this approach, one tries to assign to each game in cooperative form a unique vector of payoffs, called the value. The ith component of the value vector is considered as a measure of the value or power of the ith player in the game.
- Alternatively, the value vector may be thought of as an arbitration outcome of the game decided upon by some fair and impartial arbiter. The central "value concept" in game theory is the one proposed by Shapley in 1953.

A value function, Φ, is function that assigns to each possible characteristic function of a n-person game, v, an n-tuple of real numbers

$$\Phi(v) = (\Phi_1(v), \Phi_2(v), \dots, \Phi_n(v))$$

Here $\Phi_i(v)$ represents the worth or value of player i in the game with characteristic v. The Shapley axioms are:

1. *Efficiency.* $\sum_{i \in N} \Phi_i(v) = v(N)$ (group rationality).
2. *Symmetry.* If i and j satisfy $v(S \cup \{i\}) = v(S \cup \{j\})$, for every coalition S which does not contain i nor j, then $\Phi_i(v) = \Phi_j(v)$.
3. *Dummy Player Axiom.* If i is such that $v(S) = v(S \cup \{i\})$ for every coalition S not containing i, then $\Phi_i(v) = 0$.
4. *Additivity.* If v_1 and v_2 are characteristic functions, then $\Phi(v_1 + v_2) = \Phi(v_1) + \Phi(v_2)$.

- Axiom 2 says that if the characteristic function is symmetric in players i and j, then the values assigned to i and j should be equal.
- Axiom 4 says that the arbitrated value of two games played at the same time should be the sum of the arbitrated values of the games if they are played at different times.
- For a given coalition S, let ν_S be the following characteristic function, for all $T \subseteq N$,

$$\nu_S(T) = \begin{cases} 1, & S \subseteq T, \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Lemma

Any characteristic function v may be written as a weighted sum of characteristic functions of the form (1).

Theorem

The Shapley value may be given by

$$\Phi_i(v) = \sum_{S \subseteq N, i \in S} \frac{(|S|-1)!(n-|S|)!}{n!} \left(v(S) - v(S \setminus \{i\})\right). \qquad (2)$$

Theorem

There exists an unique function Φ satisfying the Shapley axioms.

Remarks.

- $[v(S) - v(S \setminus \{i\})]$ is the player i's marginal contribution to coalition S. For this formula the usual interpretation is the following: suppose the players enters a room in some order and that all $n!$ orderings of the players in N are equally likely. Then $\Phi_i(v)$ is the expected marginal contribution made by player i as he enters the room.
- To see this, consider any coalition S which contains player i, the probability that, when player i enters the room, he finds precisely the players from $S \setminus \{i\}$ already there is $\dfrac{(|S|-1)!(n-|S|)!}{n!}$.

First Method

- first (recursively) compute $c_{v,T} = v(T) - \sum_{T' \subsetneq T} c_{v,T'}, \forall T \subseteq N$;
- second use the formula $\Phi_i(v) = \sum_{i \in S \subseteq N} \frac{c_{v,S}}{|S|}, i = \overline{1,n}$.

Consider again the three-person game having the payoffs ($v(\emptyset) = 0$):

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	1	0	1	4	3	5	8

We find that $c_{v,\{1\}} = v(\{1\}) = 1$, $c_{v,\{2\}} = v(\{2\}) = 0$, and $c_{v,\{3\}} = v(\{3\}) = 1$; $c_{v,\{1,2\}} = v(\{1,2\}) - c_{v,\{1\}} - c_{v,\{2\}} = 4 - 1 - 0 = 3$, $c_{v,\{2,3\}} = v(\{2,3\}) - c_{v,\{2\}} - c_{v,\{3\}} = 5 - 0 - 1 = 4$, $c_{v,\{1,3\}} = v(\{1,3\}) - c_{v,\{1\}} - c_{v,\{3\}} = 3 - 1 - 1 = 1$.

Then,

$$c_{v,N} = v(N) - c_{v,\{1,2\}} - c_{v,\{2,3\}} - c_{v,\{1,3\}} - c_{v,\{1\}} - c_{v,\{2\}} - c_{v,\{3\}} =$$

$$8 - 3 - 1 - 4 - 1 - 0 - 1 = -2.$$

Now $v = \nu_{\{1\}} + \nu_{\{3\}} + 3\nu_{\{1,2\}} + \nu_{\{1,3\}} + 4\nu_{\{2,3\}} - 2\nu_{\{1,2,3\}}$, thus

$$\Phi_1(v) = 1 + \frac{3}{2} + \frac{1}{2} - \frac{2}{3} = \frac{7}{3}, \Phi_2(v) = \frac{3}{2} + \frac{4}{2} - \frac{2}{3} = \frac{17}{6},$$

$$\Phi_3(v) = 1 + \frac{4}{2} + \frac{1}{2} - \frac{2}{3} = \frac{17}{6}.$$

The Shapley value is $\Phi = \left(\frac{7}{3}, \frac{17}{6}, \frac{17}{6}\right)$, which is in the core of the game (see the corresponding section).

Second Method

We can use formula (2); we illustrate this method for $\Phi_1(v)$:

- the probability that player 1 enters first is $\frac{0! \cdot 2!}{3!} = \frac{1}{3}$, and his marginal contribution is $v(\{1\}) - v(\varnothing) = 1$;
- the probability that player 1 enters second and finds player 2 in the room is $\frac{1! \cdot 1!}{3!} = \frac{1}{6}$, $v(\{1,2\}) - v(\{2\}) = 4 - 0 = 4$;
- the probability that player 1 enters second and finds player 3 in the room is $\frac{1! \cdot 1!}{3!} = \frac{1}{6}$, $v(\{1,3\}) - v(\{3\}) = 3 - 1 = 2$;
- the probability that player 1 enters third and find players 2 and 3 in the room is $\frac{2! \cdot 0!}{3!} = \frac{1}{3}$, $v(\{1,2,3\}) - v(\{2,3\}) = 8 - 3 = 5$;

Thus $\Phi_1(v) = \frac{1}{3} \cdot 1 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 2 + \frac{1}{3} \cdot 5 = \frac{14}{6}$.

We try to find an imputation which minimizes the worst inequity.

Definition

The excess *of a coalition relative to an imputation* x *is*

$$e(\mathrm{x}, S) = v(S) - \sum_{i \in S} x_i.$$

The above excess measures the amount by which the coalition S falls short of its potential $v(S)$ in the allocation x. It clearly follows that an imputation x is in the core if and only if all of its excesses are non-positive.

Define O(x) the vector of excesses (for imputation x) arranged in decreasing order; over the set of vectors O(x) we use the lexicographic order: $u <_L v$ iff for the first component i in which u and v differ, $u_i < v_i$.

Definition

Let $X = \left\{ \mathrm{x} \in \mathbb{R}^n \ : \ \sum_{i=1}^{n} x_i = v(N), x_i \geqslant v(\{i\}), \forall 1 \leqslant i \leqslant n \right\}$ *be the set of imputations. A vector* $\nu \in X$ *is a* nucleolus *if for every* $\mathrm{x} \in X$ *we have* $\mathrm{O}(\nu) \leqslant_L \mathrm{O}(\mathrm{x})$*.*

That is the nucleolus is an imputation x that minimizes O(x) in the lexicographic ordering. The following result enumerates the main properties of nucleolus.

Theorem

The nucleolus of a game in cooperative form exists and is unique. The nucleolus is group rational, individually rational, and satisfies the symmetry axiom and the dummy axiom. If the core is not empty, the nucleolus is in the core.

The Glove Game

Story. Assume there are three players, 1, 2, 3. Players 1 and 2 each possess a right-hand glove, while player 3 has a left-hand glove. A pair of gloves has worth 1. The players cooperate in order to generate a profit.

Model. $v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 1$; $v(S) = 0$ otherwise.

Solution. The core: $\{(0,0,1)\}$; the Shapley value: $(2/3, 1/6, 1/6)$.

A Permutation Game

Story. Mr. Adams, Mrs. Benson, and Mr. Cooper have appointments with the dentist on Monday, Tuesday, and Wednesday, respectively. This schedule not necessarily matches their preferences, due to different urgencies and other factors. Their preferences (expressed in numbers) are as follows:

	Mon	Tue	Wed
Adams	2	4	8
Benson	10	5	2
Cooper	10	6	4

Model. This situation gives rise to a game in which the coalitions can gain by reshuffling their appointments. A complete description of the resulting game is:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	2	5	4	14	18	9	24

Solution. The core of this game is the convex hull of the vectors $(15, 5, 4)$, $(14, 6, 4)$, $(8, 6, 10)$, and $(9, 5, 10)$. The Shapley value is the vector $(9.5, 6.5, 8)$.

A Voting Game

Story. The United Nations Security Council consists of 5 permanent members (United States, Russia, Britain, France, and China) and 10 other members. Motions must be approved by nine members, including all the permanent members.

Model. This situation gives rise to a 15-person game so-called voting game (N, v) with $v(S) = 1$ if the coalition S contains the five permanent members and at least four non-permanent members, and $v(S) = 0$ otherwise.

Solutions. A solution to such a voting game is interpreted as representing the power of a player, rather than payoff. Such games are called *simple games*. Coalitions with worth equal to 1 are called *winning*, the other coalitions are called *losing*.

$$\Phi_i(v) = \sum_{S \text{ winning }, S\setminus\{i\} \text{ losing}} \frac{(|S|-1)!(n-|S|)!}{n!}.$$

Weighted voting games. There is a large class of simple games called weighted voting games. These games are defined by a characteristic function of the form

$$v(S) = \begin{cases} 1, & \sum_{i\in S} w_i > q \\ 0, & \sum_{i\in S} w_i \leqslant q \end{cases},$$

for some non-negative numbers w_i called *weights*, and some positive number q called the *quota*. If $q = \frac{1}{2}\sum_{i\in N} w_i$, this is called a *weighted majority game*.

Example: Consider the game with players 1, 2, 3, and 4, having 10, 20, 30, and 40 shares of stock respectively, in a corporation. Decisions require approval by a majority (more than 50%) of the shares. This is a weighted majority game with weights $w_1 = 10$, $w_2 = 20$, $w_3 = 30$ and $w_4 = 40$ and with quota $q = 50$.

Exercise: Find the core and the Shapley value for this game.

Hint: The winning coalitions are $\{2,4\}$, $\{3,4\}$, $\{1,2,3\}$, and all supersets containing one of these sets.

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