Homework 1 - november 6, 2026

Deadline and delivery: 08:00 am, november 13, 2026.

- **The homework can be solved by teams of at most four students.**
- **The solutions will be drafted in LaTeX or in a word processor like Word, Writer etc. LaTeX (correctly) written solutions will receive a bonus of 2 points.**
- **Don't write again the texts of the problems! Each problem needs at most one page for its solution.**
- **ANY COPIED SOLUTION FOR A PROBLEM FROM BELOW WILL BE PENALISED WITH 2 POINTS. SEMINAR INSTRUCTORS HAVE THE RIGHT TO VERIFY THE ORIGINALITY AND THE AUTHENTICITY OF THE SOLUTIONS.**
- **The solutions will be uploaded until 08:00 am, November 13, 2026. See the page of the course for details.**
- **Build an archive containing: the source (a .doc, .odt or .tex file) and the .pdf file both with the following name format:**

  AG_Name1_group_Name2_group_... Homework1.tex (or .doc, .odt)

  AG_Name1_group_Name2_group_... Homework1.pdf

1. Let $G = (V, E)$ be a graph with $V = \{x_1, x_2, \ldots, x_n\}$ such that $d_G(x_1) \geqslant d_G(x_2) \geqslant \ldots \geqslant d_G(x_n)$. Prove that

(a) $d_G(x_1) \leqslant |\{x_i \,:\, d_G(x_i) > 0, 2 \leqslant i \leqslant n\}| \leqslant n - 1$. When both inequalities become equalities?

(b) $d_G(x_1) + d_G(x_2) \leqslant 2 + \sum_{j=3}^{n} \min\{d_G(x_j), 2\}$.

(1 + 2 = 3 points)

2. Let $G = (V, E)$ be a 2-edge-connected graph, $s, t \in V$, $s \neq t$, and $c : E \to \mathbb{R}_+$ a cost function on its edges. Let T be a SP tree rooted at s (in such a tree each path starting in s is a shortest path; see also exercise 9 from lecture 3) and P be the shortest st-path belonging to this tree with its vertices (given in the traversal order from s to t) being $u_0 = s, u_1, \ldots, u_p = t$. Choose $0 \leqslant i \leqslant p - 1$ and let V_i^1 and V_i^2 be the sets of vertices of the connected components of $T - u_i u_{i+1}$.

Prove that there exists a st-shortest path in $G - u_i u_{i+1}$ containing exactly one edge in cross between V_i^1 and V_i^2.

(3 points)

3. Let $G = (V, E)$ be a 2-edge-connected graph, $c : E \to \mathbb{R}$ an injective cost function on its edges, and T^* be a minimum spanning tree of G with respect to c. A **neighbor** of T^* is any spanning tree T of G such that $T = T^* - e^* + e$ for certain edges $e^* \in E(T^*)$ and $e \in E(T) \setminus E(T^*)$.

Devise an algorithm for finding a minimum cost spanning tree among all neighbors of T^*.

(2 points)