

Graph Algorithms - Seminar 1

September 30 - October 3, 2025

Exercise 1.

A professor wrote p exercises and wants to assign them to n students.

- (a) In how many ways this can be done? (One exercise to each student.)
- (b) Same question, if he wants to give them different exercises.
- (c) But if he wants to give two different exercises to each of its students? (Different students can share at most one common exercise.)

Exercise 1'.

A professor wrote p exercises and wants to assign them to n students. In how many ways this can be done if a student can receive any number of exercises, including 0 (the lucky ones!), but different students cannot receive a same exercise?

Discuss the relation between p and n (on both exercises).

Exercise 2. Let $f : \mathbb{N}^* \rightarrow \mathbb{N}^*$ be a function. We define the class of complexity

$$\mathcal{O}(f(n)) = \{g : \mathbb{N}^* \rightarrow \mathbb{N}^* : \exists a, b, n_0 \in \mathbb{N}^* \text{ s. t. } g(n) \leq a \cdot f(n) + b, \forall n \geq n_0\}$$

(Note that $a = a_g, b = b_g, n_0 = n_{0,g}$.) By (abuse of notation) convention we write $g(n) = \mathcal{O}(f(n))$ instead of $g(n) \in \mathcal{O}(f(n))$.

(a) Prove that one can skip b in the above definition.

(b) Prove that if $\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} \in \mathbb{R}$, then $g(n) = \mathcal{O}(f(n))$

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Exercise 3. Prove that

(a) $\sum_{k=1}^n k = \mathcal{O}(n^2)$ and $\sum_{k=1}^n k^2 = \mathcal{O}(n^3)$;

(b) $\log_a n = \mathcal{O}(\log_b n)$ and $\sum_{k=1}^n \frac{1}{k} = \mathcal{O}(\log n)$;

Exercise 4. Legend has it that Josephus Flavius wouldn't have lived to become famous without his mathematical talents. During the Jewish-Roman war, he was among a band of 41 Jewish rebels trapped in a cave by the Romans. Preferring suicide to capture, the rebels decided to form a circle and, proceeding around it, to kill every third remaining person until no one was left. But Josephus, along with an unindicted co-conspirator, wanted none of this suicide nonsense; so he quickly calculated where he and his friend should stand in the vicious circle.

In our variation, we start with n people numbered from 1 to n around a circle, and we eliminate every second remaining person until only one survives. Let j_n be the number of the survivor.

- Find j_1, j_2, \dots, j_{10} .
- Write a recursive function that computes j_n .
- Can you give a formula for j_n ? (Hint: look what happens when n becomes a power of 2.)

Exercise 5.

What is the maximum number, r_n , of regions defined by n lines in the plane?

Exercise 6. Given a set M of $n \geq 1$ elements, a partition of M is a family of subsets of M , $\mathcal{P} = \{M_1, M_2, \dots, M_k\}$, astfel încât $M_i \neq \emptyset, \forall i$, $M_i \cap M_j = \emptyset, \forall i \neq j$ and $\bigcup_{i=1}^n M_i = M$. If \mathcal{P} and \mathcal{P}' are two partitions we say that \mathcal{P} is a **refinement** of \mathcal{P}' if for every subset $M_i \in \mathcal{P}$ there exists a subset $M'_j \in \mathcal{P}'$ such that $M_i \subseteq M'_j$. We consider the following decision problem.

PARTITION

Instance: $M, |M| = n \in \mathbb{N}^*$ and two partitions $\mathcal{P}, \mathcal{P}'$.

Question: Is \mathcal{P} a refinement of \mathcal{P}' ?

- Prove that PARTITION \in P.
- Write a recursive procedure that has to decide for two given partitions if one is a refinement of the other.