Graph Algorithms - Lecture 12

January 9, 2026

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Let $G = (V, E)$ be a graph and $\mathcal{S}$ be a surface (e.g., plane, sphere) in $\mathbb{R}^3$. An embedding of G on $\mathcal{S}$ is a graph $G' = (V', E)$ such that:

a) $G \cong G'$;

b) V' is a set of distinct points of $\mathcal{S}$;

c) Every edge $e' \in E'$ is a simple curve (Jordan arc[a]) contained in $\mathcal{S}$ joining its extremities;

d) Every point of $\mathcal{S}$ is either a vertex of G' or it is contained in at most one edge of G'.

If $\mathcal{S}$ is a plane, then G is a planar graph and G' is a plane representation of G.

If $\mathcal{S}$ is a plane and G' is a graph satisfying the above b), c) and d) constraints, then G' is a plane graph.

[a] A non-self-intersecting continuous curve.

Lemma

A graph is planar if and only if it has an embedding on a sphere.

Proof. If G is planar, let G' be a planar representation of G on the plane π. Let x be a point in π and consider a sphere $\mathcal{S}$ tangent to π in x. Let y be the diametral point of x in $\mathcal{S}$. Consider $\varphi : \pi \to \mathcal{S} \setminus \{y\}$ given by $\varphi(M) =$ the point different from y in which the line My intersects the sphere, $\forall M \in \pi$. φ is a bijection and therefore $\varphi(G')$ is an embedding of G on $\mathcal{S}$.

Conversely, if G has an embedding on a sphere $\mathcal{S}$: take y a point in $\mathcal{S}$[a], consider x, the diametral point of y on $\mathcal{S}$, construct a tangent plane π to $\mathcal{S}$ in x, and define $\psi : \mathcal{S} \setminus \{y\} \to \pi$ by $\psi(N) =$ the point in which the line yN intersects the plane π, for all $N \in \mathcal{S} \setminus \{y\}$.

[a] y is chosen s. t. $y \notin V(G) \cup E(G)$.

Planar graphs - Basic properties - Faces

The ψ-image of the embedding of G on the sphere, $\psi(G)$, is the required planar representation of G. $\square$

Let G be a plane graph. If we delete the points of G (its vertices and edges) from the plane, we get a decomposition into a finite union of maximal connected regions[a] of the plane, which are called the faces of G. Exactly one of these faces is unbounded and it is called the exterior (outer) face.

Each face is characterised by the set of edges forming its boundary.

Every cycle of G divides the plane in exactly two connected regions, hence every edge of a cycle belongs to exactly two boundaries (of two faces).

A planar graph may have different planar representations.

[a] In topological sense: any two points of a connected region can be joined by a simple curve contained in that region.

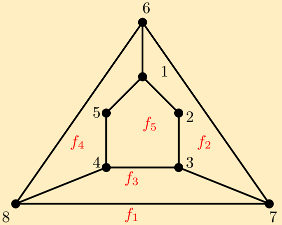
Lemma

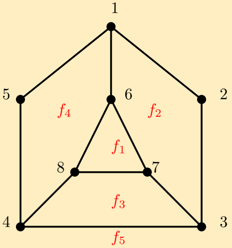
Any planar representation of a planar graph can be transformed into a (different) planar representation in which a specified face of the first one becomes the exterior face of the second.

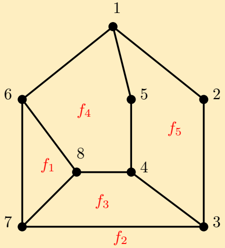
Proof. Let G' be a planar representation of G and F a face of G'. Let G^0 be an embedding of G' on a sphere and F^0 be the face of G^0 corresponding to F. Choose a point y into the interior of F^0, x its diametral point on the sphere, and π the plane tangent in x to the sphere.

$G'' = \psi(G^0)$ is a representation of G in the plane π having as its exterior face $\psi(F^0)$.

In other words the face from the sphere containing the north pole corresponds to the exterior face of the planar representation. □







Theorem

(Euler's formula) Let $G = (V, E)$ be a connected plane graph with n vertices, m edges and f faces. Then,

$$\mathrm{f} = \mathrm{m} - \mathrm{n} + 2.$$

Proof. Induction on f.

Proof cont'd. If $f = 1$, then G has no cycle and, since it is connected, it is a tree. It follows that $m = n - 1$ and the theorem holds.
In the inductive step, suppose that the theorem holds for any connected plane graph with strictly less than $f(\geqslant 2)$ faces. Let e be an edge belonging to a cycle of G (there is such cycle, since $f \geqslant 2$). Then e belongs to the boundary of exactly two faces of G. It follows that $G_1 = G - e$ is a connected plane graph having n vertices, $m - 1$ edges and $f - 1$ faces. The theorem holds for G_1, therefore $f - 1 = m - 1 - n + 2$, i. e., $f = m - n + 2$. $\square$

Remark

From an algorithmic point of view, the above theorem implies (see the next two corollaries) that any planar graph is sparse: if m is the number of edges and n is the number of vertices, then $m = \mathcal{O}(n)$.

Corollary 1

Let $G = (V, E)$ be a connected planar graph with $n \geqslant 3$ vertices and m edges. Then,

$$m \leqslant 3n - 6.$$

Proof. Let G' be a planar representation of G. If G' has only one face, then G is a tree, $m = n - 1$, and for $n \geqslant 3$ the stated inequality holds. If G' has at least two faces, then each face F of G' has in its boundary the edges of a cycle C_F (and maybe some other edges), and each such edge belongs to exactly two faces. Any cycle of G' has at least three edges, hence

$$2m \geqslant \sum_{F \text{ face of } G'} length(C_F) \geqslant \sum_{F \text{ face of } G'} 3 = 3f = 3(m - n + 2),$$

which gives the stated inequality. □

Planar graphs - Basic properties - Euler's formula

Remark

The graph K_5 is not planar (its number of vertices is $n = 5$, its number of edges is $m = 10$ and $10 > 3 \cdot 5 - 6$).

Corollary 2

Let $G = (V, E)$ be a connected bipartite planar graph with $n \geqslant 3$ vertices and $m \geqslant 3$ edges. Then,

$$m \leqslant 2n - 4.$$

Proof. Same proof as for Corollary 1, but using the fact that any cycle of G' has at least four edges. □

Remark

The graph $K_{3,3}$ is not planar (its number of vertices is $n = 6$, its number of edges is $m = 9$ and $9 > 2 \cdot 6 - 4$).

Corollary 3

If $G = (V, E)$ is a connected planar graph, then there exists $v_0 \in V$ such that

$$d_G(v_0) \leqslant 5.$$

Proof. We can suppose that G has at least two edges (to avoid trivial cases). Let G' a planar representation of G with n vertices and m edges. If we denote by n_i the number of vertices of degree i $(1 \leqslant i \leqslant n-1)$ then

$$\sum_{i=1}^{n-1} i \cdot n_i = 2m \leqslant 2(3n-6) = 6\left(\sum_i n_i\right) - 12 \Rightarrow \sum_i (i-6)n_i + 12 \leqslant 0.$$

For $i \geqslant 6$ all terms in this sum are $\geqslant 0$, thus there exists $i_0 \leqslant 5$ such that $n_{i_0} > 0$. $\square$

Let $G = (V, E)$ be a graph and $v \in V$ such that $d_G(v) = 2$ and $vw_1, vw_2 \in E$, $w_1 \neq w_2$.
Let $h(G) = (V \setminus \{v\}, E \setminus \{vw_1, vw_2\} \cup \{w_1w_2\})$.

Lemma

G is planar if and only if $h(G)$ is planar.

Proof. "$\Leftarrow$" Suppose that $h(G)$ is planar.
If $w_1w_2 \notin E$, then on the simple curve joining the points corresponding to w_1 and w_2 in a planar representation of $h(G)$ a new point is inserted corresponding to v; if $w_1w_2 \in E$ we consider a new point corresponding to v close "enough" to the curve representing w_1w_2 in one of the faces of the planar representation of $h(G)$ and "join" this new point to the points corresponding to w_1 and w_2 by simple curves disjoint from any other edges.

Proof cont'd. "⇒" Conversely, suppose that G is planar.
In its planar representation, delete the point corresponding to v and the two curves corresponding to vw_1 and vw_2 are replaced by their union; if $w_1 w_2 \in E$, then the simple curve corresponding to it is deleted. □

We denote by $h^*(G)$ the graph obtained from G by applying repeatedly the h transformation until a graph without vertices of degree two is obtained.

It follows that G is planar if and only if $h^*(G)$ is planar.

Two graphs G_1 and G_2 are homeomorphic if $h^*(G_1) \cong h^*(G_2)$.

Theorem

(Kuratowski, 1930) A graph is planar if and only if it has no subgraphs homeomorphic to K_5 or $K_{3,3}$.

Planar graphs drawing

Theorem

(Fary, 1948, independent Wagner & Stein) Every planar graph has a planar representation with all edges straight line segments (Fary's representation).

Challenge: Finding a Fary's representation with the points representing the vertices having integer coordinates and the area of the occupied surface being a polynomial in n (the number of vertices).

Theorem

(Fraysseix, Pach, Pollack, 1988) Every planar graph G with n vertices has a planar representation with vertices in points with integer coordinates in $[0, 2n-4] \times [0, n-2]$ and with all edges straight line segments.

Algorithmic proof. We will outline an $\mathcal{O}(n \log n)$ drawing.

W. l. o. g., we will assume that G is maximal planar: $\forall e \in E(\overline{G})$, $G + e$ is not planar (we add edges to G in order to make it maximal planar and when these additional edges (segments) are drawn they are invisible). Note that any face of a maximal planar graph is a triangle and has $3n - 6$ edges, where n is its number of vertices.

Lemma 1

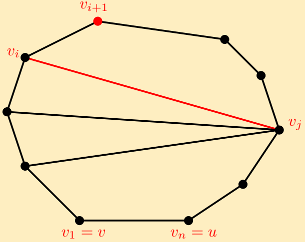
Let G be a planar graph and G' be a planar representation of G. If C' is a cycle of G' passing through the edge $uv \in E(G')$, then there exists $w \in V(C')$ such that $w \neq u, v$ and there is no interior chord of C' with one extremity in w.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of C' in a traversal of it from v to u ($v = v_1$, $u = v_n$).

Proof cont'd. If C' has no interior chords, then lemma trivially holds. Otherwise, choose the pair (i, j) such that $v_i v_j$ is an interior chord of C' and

$j - i = \min\{k - l \ : \ k > l + 1, v_k v_l \in E(G'), v_k v_l$ interior chord of $C'\}$.

Then, $w = v_{i+1}$ is not incident with an interior chord: $v_{i+1} v_p$ with $i + 1 < p < j$ cannot be an interior chord - by the choosing of the pair (i, j), and $v_{i+1} v_l$ with $l < i$ or $l > j$ is not an interior chord since it must cross $v_i v_j$. $\square$



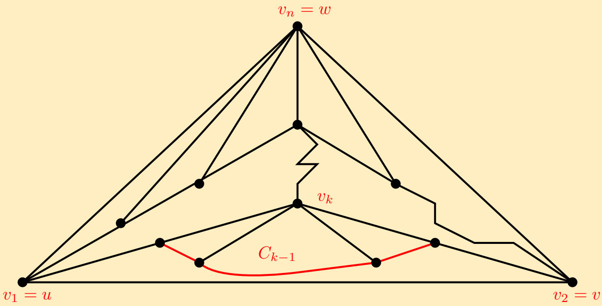
Lemma 2

Let G be a maximal planar graph with $n \geqslant 4$ vertices and G' a planar representation of G having as exterior face the triangle u, v, w. Then, there is a labeling $v_1, v_2, \ldots, v_n$ of the vertices of G' such that $v_1 = u$, $v_2 = v$, $v_n = w$ and, for every $k \in \{4, \ldots, n\}$, we have:

(i) The induced subgraph $G'_{k-1} = [\{v_1, \ldots, v_{k-1}\}]_G$ is 2-connected and its exterior face is determined by the cycle C'_{k-1} containing uv.

(ii) In the induced subgraph G'_k the vertex v_k is in the exterior face of G'_{k-1} and $N_{G'_k}(v_k) \cap \{v_1, \ldots, v_{k-1}\}$ is a path of length $\geqslant 1$ on the cycle $C'_{k-1} - uv$.

Proof. Let $v_1 = u$, $v_2 = v$, $v_n = w$, $G'_n = G$, $G'_{n-1} = G - v_n$.



Proof cont'd. Observe that $N_{G'_n}(w)$ is a cycle containing uv (after a simple sorting of $N_{G'_n}(w)$ on x-coordinate, and using the maximal planarity). It follows that i) and ii) hold for $k = n$.

If v_k has been chosen ($k \leqslant n$) then in $G'_{k-1} = G' - \{v_n, \ldots, v_k\}$, the neighbors of v_k determine a path on the cycle C'_{k-1} containing uv and forming the boundary of the exterior face of G'_{k-1}.

Proof cont'd. By Lemma 1, there exists v_{k-1} on C'_{k-1} such that v_{k-1} is not the extremity of an interior chord of C'_{k-1}.
From the construction, v_{k-1} is not adjacent with external chords of C'_{k-1} (by the maximal planarity). It follows that G'_{k-2} will contain a cycle C'_{k-2} with properties (i) and (ii). □

Proof of the Theorem (Fraysseix, Pach, Pollack). Let G be a maximal planar graph with n vertices, G' be a planar representation with vertices labeled $v_1, \ldots, v_n$ as in Lemma 2, and u, v, w its exterior face.
We will construct a Fary representation of G having as vertices points of integer coordinates.
In the step k ($\geqslant 3$) of construction, we have such a representation of G_k and the following three conditions are fulfilled:

Proof cont'd.

(1) v_1 has coordinates $(0,0)$; v_2 has coordinates $(i,0)$, $i \leqslant 2k-4$.

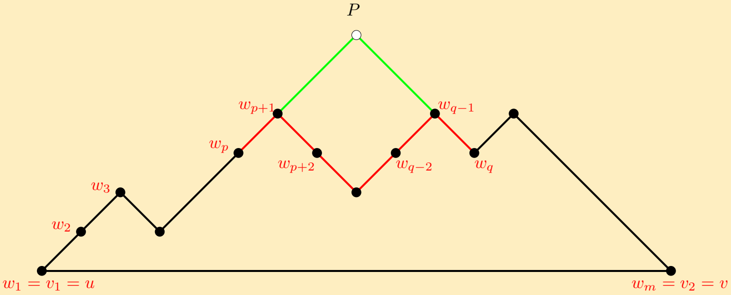
(2) If $w_1, w_2, \ldots, w_m$ are the vertices of the cycle giving the exterior face of G'_k, in a traversal from v_1 to v_2 ($w_1 = v_1$, $w_m = v_2$), then

$$x_{w_1} < x_{w_2} < \ldots < x_{w_m}.$$

(3) The edges $w_1w_2, w_2w_3, \ldots, w_{m-1}w_m$ are straight line segments parallel with one of the two bisectors of the coordinate axis.

Condition (3) implies that $\forall i < j$, the parallel through w_i to the first bisector intersects the parallel through w_j to the second bisector in a point with integer coordinates (w_i and w_j have integer coordinates).

Construction of G'_{k+1}. Let $w_p, w_{p+1}, \ldots, w_q$ be the neighbors from G'_k of v_{k+1} in G'_{k+1} ($1 \leqslant p < q \leqslant m$).



Proof cont'd. The parallel through w_p to the first bisector intersects the parallel through w_q to the second bisector in point P.

If from P we can draw the segments Pw_i, $p \leqslant i \leqslant q$ such that all are distinct, then we can take $v_{k+1} = P$ to obtain the Fary representation of G_{k+1} with all vertices having integer coordinates, satisfying the conditions (1) - (3).

If the segment $w_p w_{p+1}$ is parallel with the first bisector, then we translate with 1 to the right all vertices of G'_k having the x-coordinate $\geqslant x_{w_{p+1}}$. Make another translation with 1 of all vertices of G'_k having the x-coordinate $\geqslant x_{w_q}$.

Now, all segments $P'w_i$, for $p \leqslant i \leqslant q$, are distinct, the segments $w_i w_{i+1}$ with $i = \overline{q, m-1}$ have slopes ± 1 and also $w_p P'$ and $P' w_q$ have slopes ± 1 (where P' is the intersection of the parallel to the first bisector through w_p with the parallel to the second bisector through w_q).

Take $v_{k+1} = P'$ and the step k of the construction is finished. □

Theorem

(Tarjan & Lipton, 1979) *Let G be a planar graph with n vertices. There is a partition (A, B, S) of $V(G)$ such that:*

- *S separates A from B in G: $G - S$ has no edges from A to B,*
- $|A| \leqslant (2/3)n$, $|B| \leqslant (2/3)n$,
- $|S| \leqslant 4\sqrt{n}$.

This partition can be found in $\mathcal{O}(n)$ time.

Proof idea. Let G be a connected plane graph. Perform a bfs traversal from some vertex s, labeling each vertex v by its level in the bfs tree obtained. Let $L(t)$, the set of all vertices on the level t, for $0 \leqslant t \leqslant l+1$. The last level $L(l+1)$ is empty - for technical reasons (the last level is in fact l).

Proof cont'd. Each internal level is a separator in G (we have edges only between consecutive levels). Let t_1 be the middle level, that is, the level which contains the $\lfloor n/2 \rfloor$-th vertex encountered in the traversal. The set $L(t_1)$ satisfies:

$$\left|\bigcup_{t<t_1} L(T)\right| < \frac{n}{2} \text{ and } \left|\bigcup_{t>t_1} L(T)\right| < \frac{n}{2}.$$

If $|L(t_1)| \leqslant 4\sqrt{n}$, the theorem holds.

Lemma

There are levels $t_0 \leqslant t_1$ and $t_2 \geqslant t_1$ such that $|L(t_0)| \leqslant \sqrt{n}$, $|L(t_2)| \leqslant \sqrt{n}$ and $t_2 - t_0 \leqslant \sqrt{n}$.

Proof. Take t_0 the maximum integer satisfying $t \leqslant t_1$ and $|L(t)| \leqslant \sqrt{n}$ (there is such a level since $|L(0)| = 1$). There exists t_2 the minimum integer satisfying $t > t_1$ and $|L(t_2)| \leqslant \sqrt{n}$ (note that $|L(l+1)| = 0$). Any level between t_0 and t_2 has more than $\sqrt{n}$ vertices, therefore the number of these levels is less than $\sqrt{n}$ (otherwise, the number of vertices would be $> n$). □

Proof cont'd (Separator's Theorem). Let

$$C = \bigcup_{t<t_0} L(t), D = \bigcup_{t_0<t<t_2} L(t), E = \bigcup_{t>t_2} L(t).$$

- $|D| \leqslant (2/3)n$. The theorem holds with $S = L(t_0) \cup L(t_2)$, A the set with a maximum cardinality among C, D, E and B the union of the remaining two sets (C and E have at most $n/2$ elements).

- $n_1 = |D| > (2/3)n$. If we could find a separator of type $1/3 \leftrightarrow 2/3$ for D with at most $2\sqrt{n}$ vertices, then we add it to $L(t_0) \cup L(t_2)$ in order to obtain a separator of cardinality at most $4\sqrt{n}$, take as A the union of the set of maximum cardinality between C and E with the small part remained in D, and take as B the union of other two remaining sets.

The separator for (the graph induced by) D can be constructed as follows: delete all the vertices of G which are not from D, excepting s which is joined with all vertices of the level $t_0 + 1$. The graph obtained is denoted by D and is planar and connected. It has a spanning tree of diameter at most $2\sqrt{n}$ (any vertex is reached from s by a path of length at most $\sqrt{n}$, as we proved in the above Lemma). This tree is dfs traversed in order to obtain the desired separator. Details (very nice) are omitted. $\square$

Let us consider the problem of deciding if a given planar graph has a (vertex) 3-coloring, which is a NP-complete problem.

For a graph G with a small number of vertices (for a constant number c, we can verify all $\mathcal{O}(3^c) = \mathcal{O}(1)$ functions from $V(G)$ to $\{1, 2, 3\}$) we can easily decide if it has a 3-coloring.

For planar graphs with the number n of vertices $> c$, we build in $\mathcal{O}(n)$ the partition (A, B, S) of its vertex set, with $|A|, |B| \leqslant (2n/3)$ and $|S| \leqslant 4\sqrt{n}$ like in the above theorem.

For each of the $3^{|S|} = 2^{\mathcal{O}(\sqrt{n})}$ possible functions from S to $\{1, 2, 3\}$ we test if there exists a 3-coloring of the subgraph induced by S and if it can be extended to a 3-coloring of the subgraph induced by $A \cup S$ in G and also to a 3-coloring of the subgraph induced by $B \cup S$ in G (recursively).

The running time $T(n)$, of this algorithm, satisfies the recursion

$$T(n) = \begin{cases} \mathcal{O}(1), & \text{if } n \leqslant c, \\ \mathcal{O}(n) + 2^{\mathcal{O}(\sqrt{n})}\left(\mathcal{O}(\sqrt{n}) + 2\,T(2n/3)\right), & \text{if } n > c. \end{cases}$$

It follows that $T(n) = 2^{\mathcal{O}(\sqrt{n})}$ (it is possible that the constants behind the notation $\mathcal{O}(\cdot)$ to be very large).

Exercise 1. Let $G = (V, E)$ be a plane connected graph on n vertices and m edges.

(a) If the length of the cycle on the boundary of each face is at least $k \geqslant 3$ for an integer k, then $m \leqslant \frac{k(n-2)}{k-2}$.

(b) Prove that the Petersen's graph is not planar.

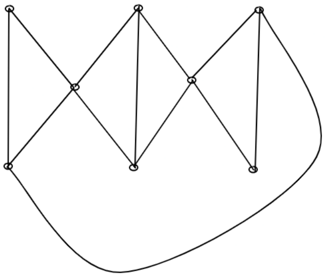
Exercise 2. Let $G = (V, E)$ be a plane graph with n vertices, m edges, and p connected components. Find a formula for the number of its faces in terms of n, m, and p.

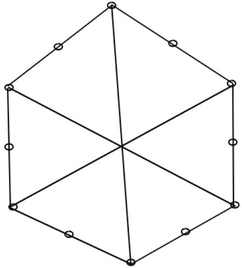
Exercise 3. Which of the following graphs have the property that the removal of any vertex would result in a planar graph?

$$K_5, K_6, K_{4,3}, K_{3,3}, \text{ Petersen's graph.}$$

Exercise 4. The crossing number, $cr(\cdot)$, of a graph is the minimum number of crossings occuring when the graph is drawn in the plane (under the assumption that three edges cannot intersect at the same non-vertex point). Find the crossing number of the following graphs: $K_{3,3}$, K_5, K_6, and the Petersen's graph.

Exercise 5. Use Kuratowski's theorem to find out which of the following graphs are planar:

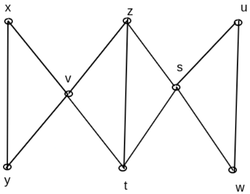


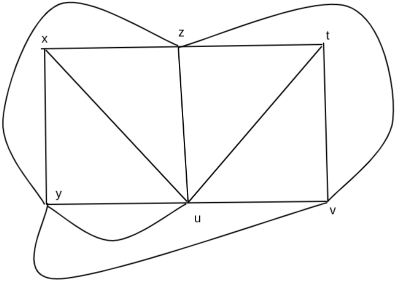


Exercise 6. Let G be a plane (multi-) graph, define a multi-graph, G^*:

- to each face f of G will correspond a vertex f^* of G^*;
- to each edge e of G will correspond an edge e^* of G^*.
- two vertices f_1^* and f_2^* are joined by an edge e^* if and only if the faces f_1 and f_2 share the edge e in their boundaries.

Draw the duals of the following planar graphs:



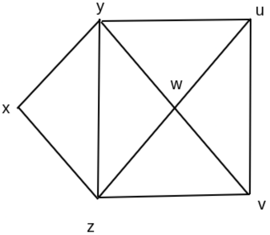


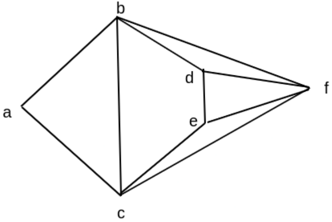
Exercise 7.

(a) Prove that the dual of a plane graph is planar.

(b) If G is a connected plane graph, then $G^{**} \cong G$.

Exercise 8. Prove that the following two plane graphs are isomorphic but their duals are not.





Exercise 9. Let G be a connected plane graph and G^* its dual.

(a) If T is a spanning tree of G, then the edges of G^* which do not correspond to $E(T)$ are the edges of a spanning tree of G^*.

(b) The number of spanning trees in G equals the number of spanning trees in its dual, G^*.

Exercise 10. Let G be a plane graph with triangular faces; color at random with three colors all of its vertices. Prove that the number of faces receiving all three colors is even.

Exercise 11*. Let G be a plane graph having all degrees even. Prove that we can color its faces with two colors such that any two faces with a common edge in their boundaries have different colors.

Exercise 12*. Let G be a plane graph with triangular faces ($|G| \geqslant 4$). Prove that its dual, G^* is 2-edge connected and 3-regular (as a consequence G^* has a perfect matching).