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Course information

C. Croitoru - Graph Algorithms * C. Croitoru - Graph Algorithms * C. Croitoru - Graph Algorithms
* C. Croitoru - Graph Algorithms * C. Croitoru - Graph Algorithms * C. Croitoru - Graph

Course page:

<https://edu.info.uaic.ro/algorithmica-grafuri>

Objectives:

The lectures will cover basic topics in Algorithmic Graph Theory. Accumulated knowledge will be applied in designing efficient algorithms for combinatorial optimization problems.

Lectures: E. F. Olariu (halfyear A and E), C. Frăsinaru (halfyear B).

- The lecture notes will be posted as .pdf files before each lecture.
- Each of these lectures notes will contain the exercises for the corresponding seminars.

Seminars: E. F. Olariu, A. C. Frăsinaru, P. Diac, S. Amihăesei, and A. Ioniță

- All lectures and seminars will take place [onsite](#).
- Each seminar will contain a short test about past lectures and seminars knowledge.
- During the seminars we will discuss solutions to proposed exercises (which can be found in the corresponding lecture file).
- Each seminar debates three or four exercises (posted in advance in the lecture notes) in order to deepen the subjects introduced in the lectures.
- Students are encouraged to find and present original solutions to these problems.

Scores:

- **Seminar:** tests (max. $2 \times 10 = 20\text{p}$) and activity^a (max. 22p^b) - max. 42 points.
- **Homeworks:** three exercise sheets, 16p each - max. 48 points. (The homeworks can be solved in teams of at most four students, each being leader/responsible for at least one problem from the set.)
- **Written final test (compulsory):** six exercises (10p each) +10p the basis gives a maximum of 70 points.

From a maximum of 90p for seminar activity and homeworks the threshold for taking the written final test is of 30 points.

From a maximum of 160p the threshold for promoting the course is of 80 points.

^aTwo correct and pertinent answers = 1 point.

^bApproximatively.

Grading:

- **Seminar.** Each test is a question whose not too long answer needs a justification.
- **Homeworks.** The solutions can be worked in teams of up to four students (each being leader/responsible for at least one problem from the corresponding set). Solutions will be written in **LaTeX** or Word (Write) and electronically sent (both the source and the corresponding .pdf). 2 points bonus for each homework written in **LaTeX**. The solution will be checked using an anti-plagiarism tool, the grades of the students which will copy will be penalised (for each copied problem the student will receive -2 points only).
- **Written final test (compulsory).** The students cannot use course related materials during the test.

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Outline of the course:

- Vocabulary of Graph Theory.
- Path problems: graph traversal, shortest-paths, connectivity.
- Minimum spanning trees: union-find, amortized complexity.
- Matching theory.
- Flows.
- Polynomial time reductions between decision problems on graphs.
- Approaching NP-difficult problems.
- Planar graphs.
- Tree-decomposition.

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- **Graph colorings:** coloring maps (faces of planar graphs), lectures/seminars/timetabling, exam scheduling for university departments, mobile radio frequency allocation, memory register allocation.

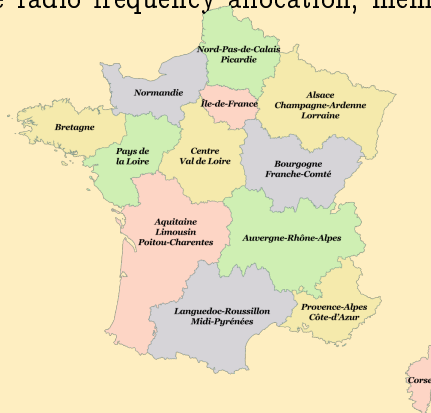


Figure: France regions from 2016

Graph Theory applications

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- **Matchings:** assignment problems, in computational chemistry and mathematical chemistry studies on organic materials.

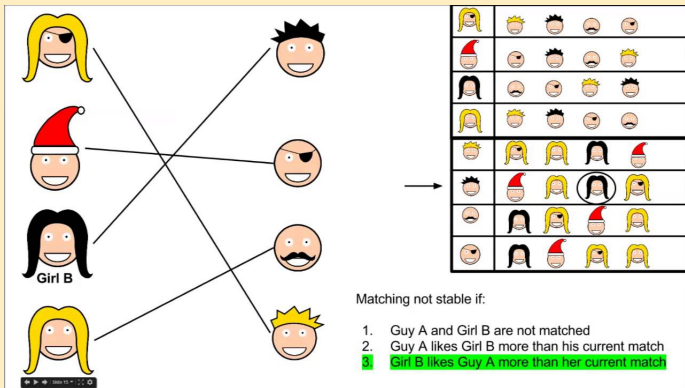


Figure: Gale Shapley algorithm

Applications in Computer Science

- In database management, **graph database** uses graph-structure data for storing and querying.
- **Graph rewriting systems** are used in **software verification**.
- **Quantum computation**.
- **Modeling the web documents as graphs** and clustering of web documents.
- Approximation of data and data compression.
- Modeling the sensor network as graphs (using Voronoi diagrams).
- Graph Neural Networks in AI.
- And a lot more. Like the **analysis** of **social networks** in order to find characteristics like:

Graph Theory applications

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- the level of **connectivity**, the density;
- the influence of users within the network (**centrality**, the **social networking potential**);
- the level of **segmentation**: measuring the clusterization;
- robustness or structural stability of the network.

The network analysis is then used for

- data mining and aggregation, marketing;
- behavior analysis and network prediction;
- intelligence and security analysis (surveillance of terrorism and organized crime) etc.

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Graph Theory applications

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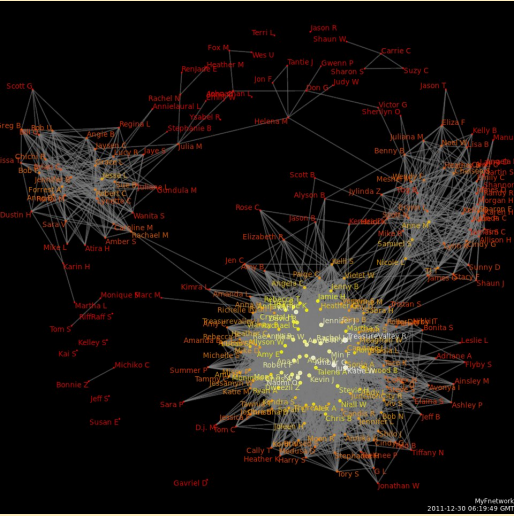


Figure: A Facebook network of a few users

Definition 2

If $G = (V, E)$ and $e = uv = vu = \{u, v\} \in E$ is an edge of G , then we say that

- e **connects** (or **links**) vertices u and v ; that
- vertices u and v are **adjacent** or u and v are **neighbors**;
- e is **incident** with u and v ;
- u and v are the **endpoints** (**extremities**) of e .

Neighborhood of vertex u is $N_G(u) = \{v \in V(G) : uv \in E(G)\}$.

Two edges e and f are **adjacent** if they share a common endpoint: $|e \cap f| = 1$.

Graph definition

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Let $G = (V, E)$ be a graph and $v \in V$.

- the **degree** of vertex v : $d_G(v) = \text{number of edges incident to } v$.
- v is an **isolated vertex** if $d_G(v) = 0$; v is called **pendant** (or **leaf**) if $d_G(v) = 1$.
- an useful property:

$$\sum_{v \in V} d_G(v) = 2|E|.$$

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A graph can be represented in plane as a figure consisting of a set of nodes (small geometric forms: points, circles, squares etc) corresponding to its vertices and curves which connect the vertices corresponding to the edges from the graph.

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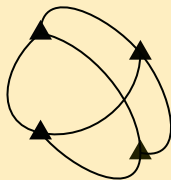
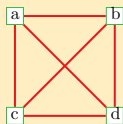
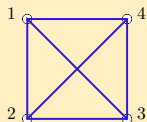
Graph definition

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We can add labels (names, numbers etc) and colors to vertices and edges obtaining better visual representations.

Below there are three visual representations of the same graph:

$$G = (\{1, 2, 3, 4\}, \{12, 13, 14, 23, 24, 34\})$$



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Definition 3

Stable set (or **independent set of vertices**) in $G = (V, E)$: $S \subseteq V$ such that $\binom{S}{2} \cap E = \emptyset$.

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In other words $S \subseteq V$ is a stable set of G if there is no edge between its vertices.

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Notation

The maximum cardinality of a stable set of G is the **stability number** (or **independence number**) of G and is denoted by $\alpha(G)$.

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Matchings

Definition 4

Matching (independent set of edges) in $G = (V, E)$: $M \subseteq E$ such that for all $e, f \in M$, if $e \neq f$, then $e \cap f = \emptyset$.

In more words, $M \subseteq E$ is a matching if each pair of its edges share no endpoint.

Notation

The maximum cardinality of a matching in G is called the **matching number (edge-independence number)** of G and is denoted by $\nu(G)$.

Matchings

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Maximum matching: Optimization problem

P2 Input: G graph.

Output: $\nu(G)$ and a witness matching M with $|M| = \nu(G)$.

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Edmonds (1965) showed that $P2 \in P$.

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Note

The problems $P1$ and $P2$ are similar: in both we are required to find a member of maximum cardinality of a family of sets concerning a given graph. What makes the difference?

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Definition 5

For $p \in \mathbb{N}^*$, a **p -coloring** of the graph $G = (V, E)$ is a map $c : V \rightarrow \{1, \dots, p\}$ such that $c(u) \neq c(v)$ for each $uv \in E$.

It is worth noting that the set of all vertices having the same given color is a stable set (it is called also a **coloring class**). Since adjacent vertices have different colors, a p -coloring is a partition of V with at most p stable sets (or **coloring classes**).

Notation

The **chromatic number** of the graph G is the least value of p such that G has a p -coloring. This parameter is denoted by $\chi(G)$. ($\chi(G) \leq |G|$ - why?)

Graph colorings

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Definition 6

For $p \in \mathbb{N}^*$, a **p -edge coloring** of the graph $G = (V, E)$ is a map $c : E \rightarrow \{1, \dots, p\}$ such that $c(e) \neq c(f)$ for all $e, f \in E$ with $|e \cap f| = 1$.

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We can note that in a p -edge coloring a set of edges with the same color is a matching. Hence, a p -edge coloring is a partition of E with at most p matchings.

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Notation

Chromatic index of the graph G : the least value of p such that G has a p -edge coloring. This parameter is denoted by $\chi'(G)$. ($\chi'(G) \leq |E(G)|$ - why?)

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Graph isomorphism

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Definition 7

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there is a bijection $\varphi : V_1 \rightarrow V_2$ such that for every two vertices $u_1, v_1 \in V_1$, u_1 and v_1 are adjacent in G_1 (i. e., $u_1 v_1 \in E_1$) if and only if $\varphi(u_1)$ and $\varphi(v_1)$ are adjacent in G_2 (i. e., $\varphi(u_1)\varphi(v_1) \in E_2$).

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In other words two graphs are isomorphic if there is a bijection between their sets of vertices which induces a bijection between their sets of edges.

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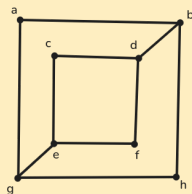
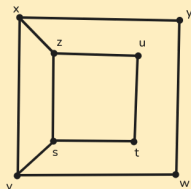
Notation

$G_1 \cong G_2$.

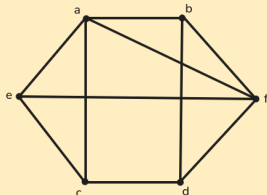
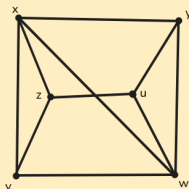
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Exercises for the 2nd seminar

Exercise 1. The following graphs are isomorphic?



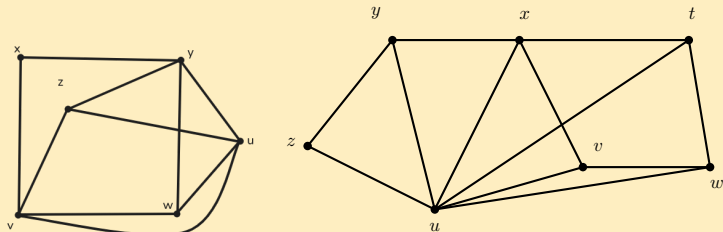
Exercise 2. The following graphs are isomorphic?



Exercises for the 2nd seminar

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Exercițiul 4. Find the stability number of the following graphs



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Exercise 5. Let $P(n) =$ "In any graph with at least n vertices there are three distinct vertices which are pairwise adjacent or pairwise non-adjacent." Prove that 6 is the least value of $n \in \mathbb{N}, n \geq 3$ for which $P(n)$ is true.

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Exercises for the 2nd seminar

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Exercise 7. Two students, **L(azy)** and **T(hinky)**, must find a particular path between two given vertices in a sparse graph G : $|E(G)| = O(|V(G)|)$. **L** judge that, since the graph is sparse, the number of paths between the two vertices is small, and a lazy solution is to generate (backtracking) all these paths and than retain the wanted one. **T** does not agree and gives the following example: let $H_n = K_2 \times P_n$ ($n \geq 1$); add to H_n two new vertices x and y each joined by two edges with one of the two pairs of adjacent vertices in H_n of degree 2. The obtained graph, G_n , is sparse but the number of paths between x and y is big. Explain to **L** this example by drawing G_n , showing its sparsity, and finding the number of paths between x and y .

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Exercise 8. An examination session must be scheduled based on the following input specifications: the set of exams is known; each student sends the list of exams to which (s)he is registered; each exam take place with all students registered to it (written exam); each student can participate to at most one exam in the same day.

Construct a graph in order to answer the following questions (by determining appropriate parameters):

- (a) What is the maximum number of daily exams?
- (b) What is the minimum number of days for the examination session?

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Exercises for the 2nd seminar

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Exercise 8'. (exercise 8 cont'd) A smart and skilled programmer is wondering if the two NP-hard problems in the above exercise can be solved in polynomial time since the graph constructed seems to belong to a very special class of graphs.

(a) Prove that for any graph, G , there is an input for the scheduling problem above such that the graph constructed is G .

The programmer suggests the following "greedy approach" to solve the second question in exercise 8: starting with day 1, a maximum number of exams are scheduled (from the set of exams not scheduled) daily, until all exams are scheduled.

b) Show that this greedy strategy can fail, by giving a counterexample.

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Exercise 9. A compiler optimization is the **register allocation technique**: the most used variables are kept in the fast processor registers in order to have them there at the moment when the compiler needs them (for certain operations made by CPU).

Design a graph that - by using its appropriate parameters - will answer to the following questions:

- (a) What is the maximum number of variables which are not needed at the same time?
- (b) What is the minimum number of registries needed?

Hint: We have two kind of objects at hand: **variables** (with their values) and **CPU operations** (or operators) which use one or more variables.

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Exercises for the 2nd seminar

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Exercise 9'. (exercise 9 cont'd) A student is wondering if the above questions (which correspond to NP-hard problems) can be answered by polynomial time algorithms since the graph constructed seems to belong to a very special class of graphs.

(a) Prove that for any graph G there is an input for the registry allocation problem such that the resulted graph is G .

The student suggests the following "greedy approach" to answer the second question in exercise 9: starting with the first step, allocate as many as possible variables to one new free register in each step, until there are no more variables.

(b) Show that this greedy approach fails, by giving a counterexample.

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